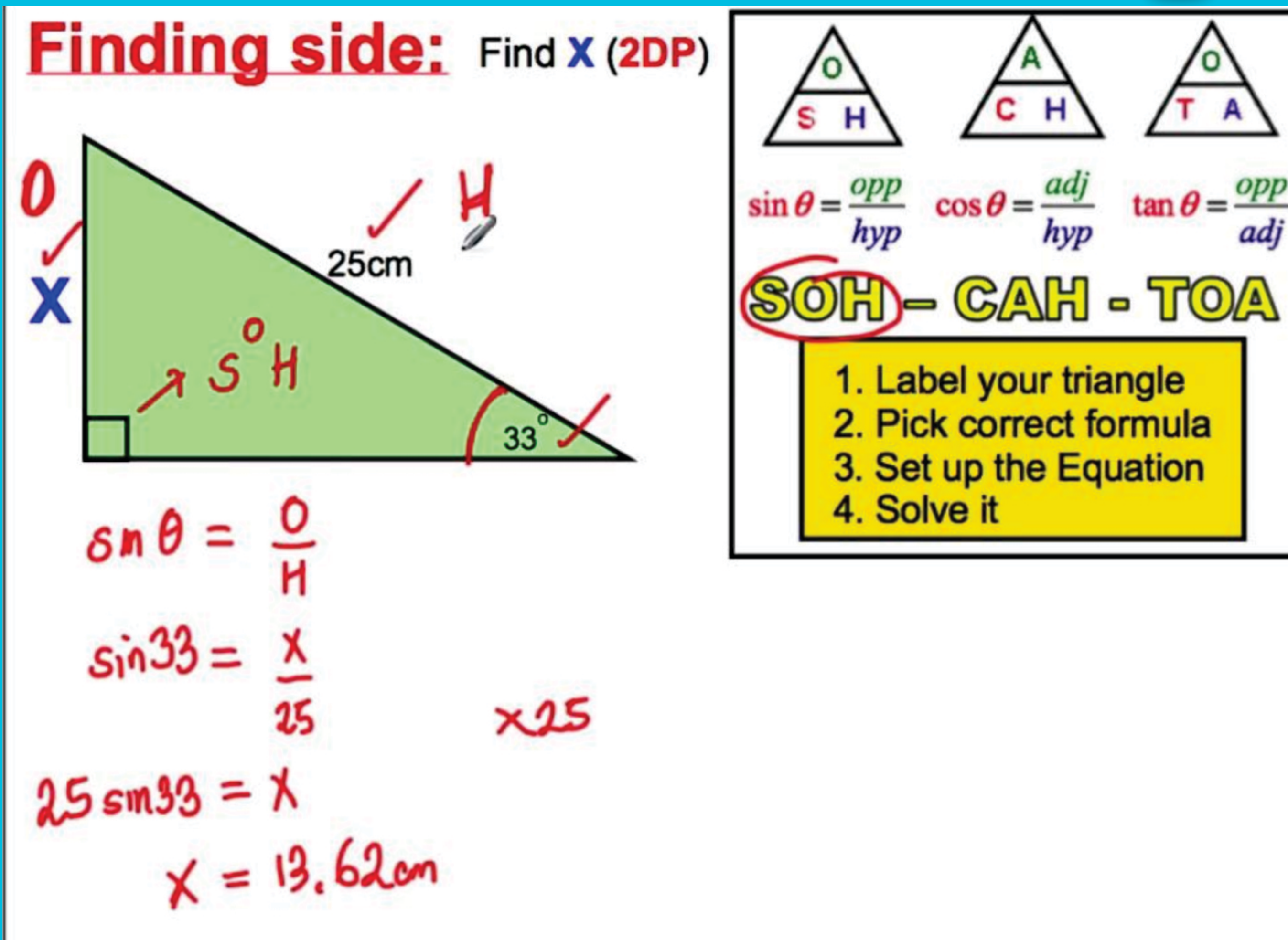
South Sudan

Secondary Mathematics 4

Student's Book



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South Sudan

SECONDARY

4

Mathematics

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NAIROBI, KENYA

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Unit 1

COMPLEX NUMBERS

Exercise 1

In pairs, work out the following problems.

1. Explain why $\sqrt{-16}$ is not a real number.
2. Add: $(2x + 5) + (3x + 7)$
3. Subtract: $(x - 1) - (4x + 8)$
4. Multiply: $9x(5x + 6)$
5. Multiply: $(2x + 3)(7x - 1)$
6. Solve: $x^2 - 24 = 0$

Some quadratic equations do not have real-number solutions because we need to find the square root of a negative number.

Example 1

Solve: $n^2 = -4$

$n = \pm\sqrt{-4}$ (No real number solutions)

Example 2

Solve: $x^2 - x + 6 = 0$

$x = \frac{1 \pm \sqrt{-23}}{2}$ (No real number solutions)

Remember, the square root of a negative number is not a real number. An expanded number system, called the complex number system, has been devised to give meaning to expressions such as $\sqrt{-4}$ and $\sqrt{-23}$.

Complex numbers

Complex numbers are numbers that consist of two parts — a real number and an imaginary number. Complex numbers are the building blocks of more intricate

mathematics, such as algebra. They can be applied to many aspects of real life, especially in electronics and electromagnetism.
The standard format for complex numbers is $a + bi$, with the real number first and the imaginary number last. i is defined as the square root of -1.

Because either part could be 0, any real number is also a complex number. Complex does not mean complicated; it means that the two types of numbers combine to form a complex, like a housing complex — a group of buildings joined together.

Real numbers are tangible values that can be plotted on a horizontal number line, such as fractions, integers or any number that you can think of. Imaginary numbers are abstract concepts that are used when you need the square root of a negative number.

The complex number system is an algebraic extension of the ordinary real numbers by the imaginary number i.

If $i = \sqrt{-1}$

$$\text{Then } i^2 = \sqrt{(-1)}^2 = -1$$

Examples

a) $\sqrt{-25} = \sqrt{25} \times \sqrt{-1}$

$= 5i$

b) $\sqrt{-81} = \sqrt{81} \times \sqrt{-1}$

$= 9i$

c) $\sqrt{-18} = \sqrt{18} \times \sqrt{-1}$

$= \sqrt{9 \times 2} \times \sqrt{-1}$

$= 3\sqrt{2}i$

d) $\sqrt{-20} = \sqrt{4 \times 5 \times -1}$

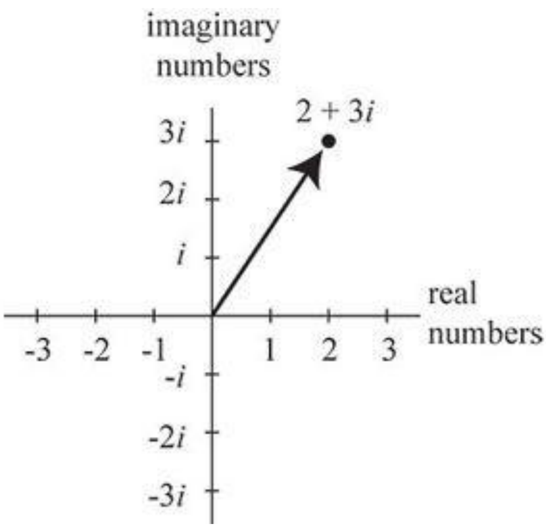
$= 2\sqrt{5}i$

e) $-\sqrt{-50} = \sqrt{25 \times 2 \times -1}$

$$= -25\sqrt{2}i$$

The Argand Diagram

We know that a real number can be represented as a point on a number line. By contrast, a complex number is represented as a point in a coordinate plane. The horizontal axis of the coordinate plane is called the real axis. The vertical axis is called the imaginary axis. The coordinate system is called the complex plane. Every complex number corresponds to a point in the complex plane and every point in the complex plane corresponds to a complex number. When we represent a complex number as a point in the complex plane, we say that we are plotting the complex number.



It is very useful to have a graphical or pictorial representation of complex numbers. For example, the complex number $z = 3+ 4i$ is represented as a point in the *xy* plane with coordinates (3,4) as shown in Figure 1. That is, the real part, 3, is plotted on the *x* axis, and the imaginary part, 4, is plotted on the *y* axis.

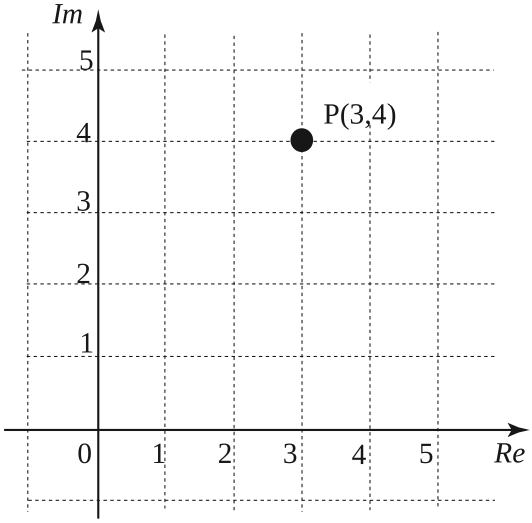


Figure 1. Argand diagram which represents the complex number 3+4i by the point P(3,4).

More generally, the complex number $z = a + ib$ is plotted as a point with coordinates (a,b) as shown in Figure 2.

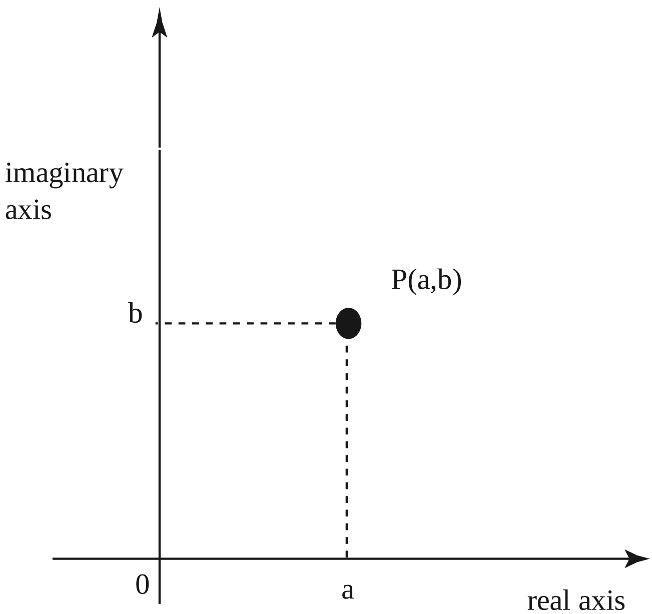


Figure 2. Argand diagram which represents the complex number $a + b$i by the point P(a,b).

Because the real part of z is plotted on the horizontal axis we often refer to this as the real axis. The imaginary part of z is plotted on the vertical axis and so we refer to this as the imaginary axis. Such a diagram is called an Argand diagram. Engineers often refer to this diagram as the complex plane.

Plot the complex numbers $z_1 = 2 + 3i$, $z_2 = -3 + 2i$, $z_3 = -3 - 2i$, $z_4 = 2 - 5i$, $z_5 = 6$, $z_6 = i$ on an Argand diagram.

Solution

The Argand diagram is shown in Figure 3.

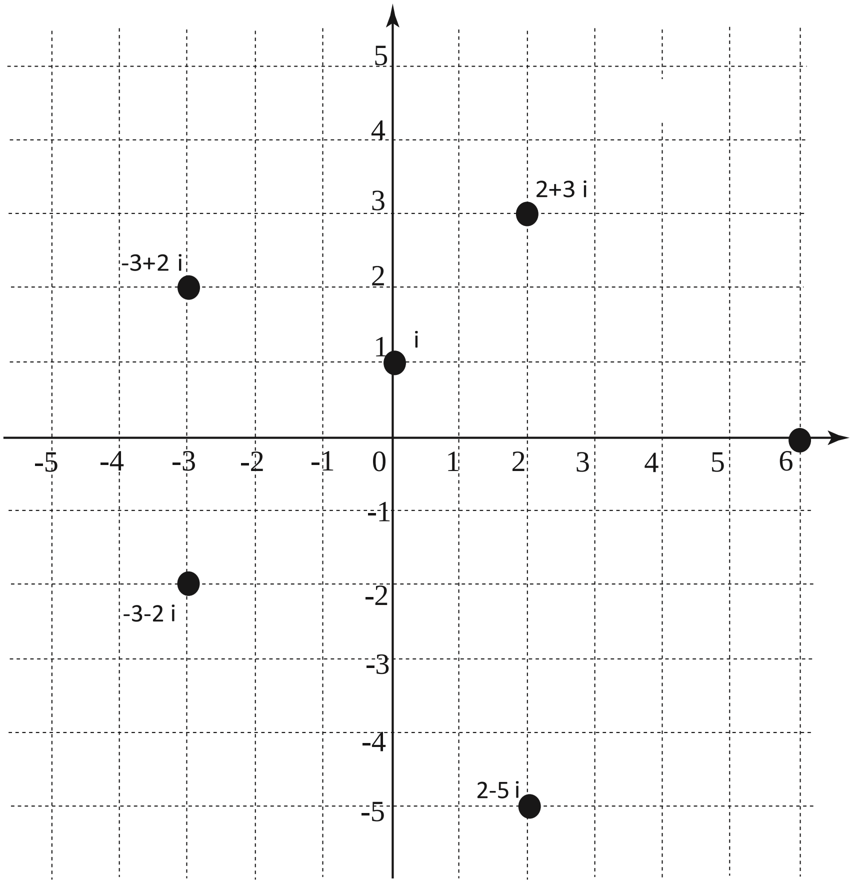


Figure 3. Argand diagram showing several complex numbers.

Note that purely real numbers lie on the real axis. Purely imaginary numbers lie on the imaginary axis.

Another observation is that complex conjugate pairs (such as $-3 + 2i$ and $-3 - 2i$) lie symmetrically about the *x* axis.

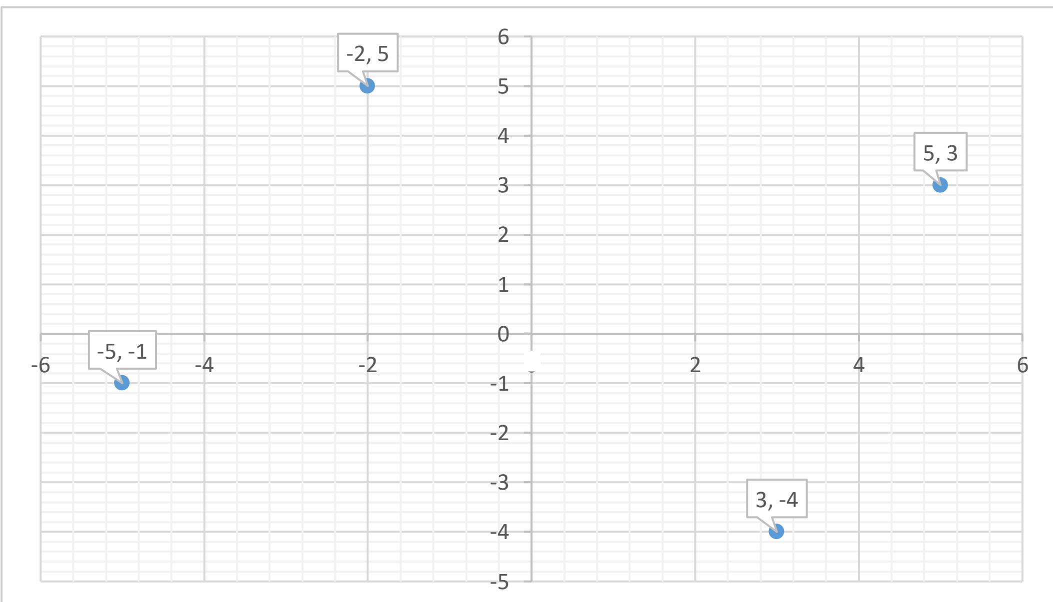
Finally, because every real number, *a* say, can be written as a complex number, *a* $+ 0i$, that is as a complex number with a zero imaginary part, it follows that all real numbers are also complex numbers. As such we see that complex numbers form an extension of the sets of numbers with which we were already familiar. Consider the complex plane below.

Example

Plot the following points on the Cartesian plane

A $= 5 + 3i$ C $= 2 + 5i$

B$= 3 - 4i$ D $= -5 - i$



Exercise 2

1. Plot the following point on the Argand diagram.

   i) A = 2 + i

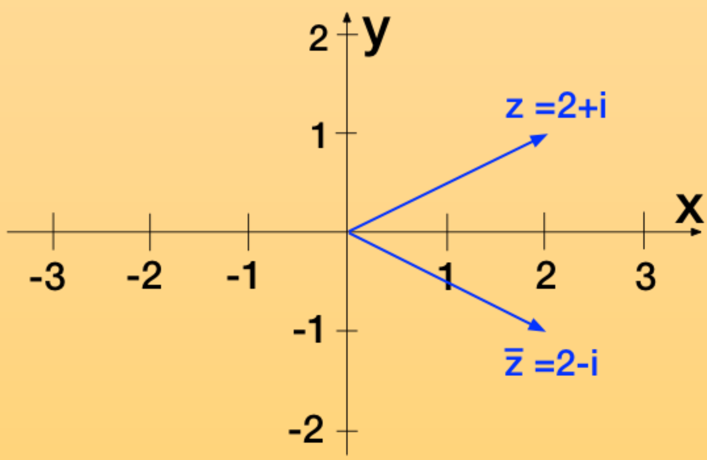
   ii) B = 2 + 3i

   iii) C = -i + 4

   iv) D = $\frac{1}{2}$ + -i

Conjugates of complex numbers

Every complex number has associated with it another complex number known as its complex conjugate. You find the complex conjugate simply by changing the sign of the imaginary part of the complex number.



Example 1

To find the complex conjugate of $4 + 7i$ we change the sign of the imaginary part. Thus the complex conjugate of $4 + 7i$ is $4 - 7i$.

Example 2

To find the complex conjugate of $1 - 3i$ we change the sign of the imaginary part. Thus the complex conjugate of $1 - 3i$ is $1 + 3i$.

Example 3

To find the complex conjugate of $-4 - 3i$ we change the sign of the imaginary part. Thus the complex conjugate of $-4 - 3i$ is $-4 + 3i$.

Operations on complex numbers

Basics of Complex Numbers

Complex numbers are created when you combine an imaginary number with a real number. This is often defined in its most basic form as $a + bi$ with "a" representing the real number and "b" the imaginary amount. In order to fully understand the basics of a complex number you need to look at the first rule of complex numbers with solving a problem such as this:

$$6 - 3i = a + bi$$

The answer is very straightforward as it highlights the main concept of complex numbers that they can only be of equal value if the imaginary and real number are the same. Therefore the solution to this problem is that $6 = x$ and $-3 = y$.

A slightly simplified but useful way to think of operations using complex numbers is that they are treated in a similar way to binomials, so like terms are combined to get to the answer. This applies when adding, subtracting and multiplying complex numbers and just requires that you remember to combine the imaginary and real parts separately.

Adding & Subtracting complex numbers

Because a complex number is a binomial — a numerical expression with two terms — arithmetic is generally done in the same way as any binomial, by combining the like terms and simplifying. For example:

$(3 + 2\mathrm{i}) + (4 - 4\mathrm{i})$
$(3 + 4) = 7$
$(2\mathrm{i} - 4\mathrm{i}) = -2\mathrm{i}$
The result is $7 - 2\mathrm{i}$.

Exercise 3

1. Work out the following

a) $(-4 + 7i) + (5 - 10i)$

b) $(4 + 12i) - (3 - 15i)$

c) $5i - (-9 + i)$

d) $(3 + 4i) - (6 - 10i)$

e) $2(-3i + 4i) - 3(5 - i)$

Multiplying Complex Numbers

The following expression is a little more complicated because two binomials are being multiplied. Once the binomials have been multiplied, simplify the expression by combining like terms.

$$(6x + 8)(4x + 2) = 6x(4x + 2) + 8(4x + 2)$$

$$= 6x(4x) + 6x(2) + 8(4x) + 8(2)$$

$$= 24x^2 + 12x + 32x + 16$$

$$= 24x^2 + 44x + 16$$

Example

Problem	Multiply and simplify $(6 + 8i)(4 + 2i)$	
	$(6 + 8i)(4 + 2i)$ $= 6(4 + 2i) + 8i(4 + 2i)$ $= 6(4) + 6(2i) + 8i(4) + 8i(2i)$ $= 24 + 12i + 32i + 16i^2$	Two binomials are being multiplied, so you need to use the Distributive Property of Multiplication.
	$= 24 + 44i + 16i^2$	Combine like terms.
	$= 24 + 44i + 16(-1)$ $= 24 + 44i - 16$ $= 8 + 44i$	Replace i^2 with -1 and simplify.
Answer	$(6 + 8i)(4 + 2i) = 8 + 44i$	

In this case, the product of two complex numbers is complex. But in the following example, the product is real, not complex. See if you can figure out why!

Task

Multiply and simplify $(6 + 8i)(6 - 8i)$

Exercise 4: Work in pairs.

Simplify.

1. $(-4i)^3$
2. $(i)(2i)(-7i)$
3. $(3i)(-2 + 6i)$
4. $(-2i)^3$
5. $(6i)(-4i)$
6. $(-8i)^2$
7. $(-3 + 5i)(-5 + 7i)$
8. $(5 - 3i)(-8 + 5i)$
9. $(5 + 5i)(-3 - 7i)$
10. $(-7 - 4i)(-6 - 6i)$

Dividing complex numbers

Division, however, becomes more complicated and requires using conjugates. Complex conjugates are pairs of complex numbers that have different signs, such as *(a + bi)* and *(a - bi)*. Multiplying complex conjugates causes the middle term to cancel out. For example:

$$(a + bi)(a - bi) = a^2 - abi + abi - (bi)^2$$

This simplifies to $a^2 - b^2(i^2) = a^2 - b^2(-1) = a^2 + b^2$

When dividing complex numbers, determine the conjugate of the denominator and multiply the numerator and denominator by the conjugate. For example,

$$(5 + 2i) \div (7 + 4i)$$

The conjugate of 7 + 4i is 7 - 4i. So, multiply the numerator and denominator by the conjugate:

$$(5 + 2i)(7 - 4i) \div (7 + 4i)(7 - 4i)$$

$$= (35 + 14i - 20i - 8i^2) \div (49 - 28i + 28i - 16i^2)$$

$$= (35 - 6i + 8) \div (49 + 16)$$

$$= (43 - 6i) \div 65$$

Example

1. Work out: $(2 - i)(3 + 4i)$

$$= (2)(i) + (2)(4i) + (-i)(-4i)$$

$$= 6 + 8i - 3i - 4i^2 = 6 + 5i + 4$$

$$= 10 + 5i$$

2. Simplify

$$\frac{3}{2i} = \frac{3}{2i} \times \frac{i}{i} = \frac{3i}{2i^2} = \frac{3i}{2(-2)} = \frac{3i}{-2} = \frac{-3}{2}i$$

3. Simplify, $\frac{3}{2+i}$

$$\frac{3}{2+i} \times \frac{2-i}{2-i} = \frac{6-3i}{4-i^2} = \frac{6-3i}{4-(-1)} = \frac{6-3i}{5}$$

Exercise 5: Work in pairs.

1. Simplify the following complex numbers.

   a) $(4 + 3i)(2 - i)$

   b) $\frac{1+2i}{3-4i}$

   c) $7i(-5 + 2i)$

   d) $(1 - 5i)(-9 + 2i)$

   e) $(1 - 8i)(1 + 8i)$

   f) $(5 + i)\,2(+6i)$

2. Verify the following.

   a) $(\sqrt{2} - i)$ - i (1 - $\sqrt{2}i$) =-2i

   b) $\frac{1+2i}{3-4i} + \frac{2-i}{5i} = -\frac{2}{5}$

   c) $\frac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2}$i

   d) $(1 - i)^2 = -2i$

Polar form of complex numbers

You have already seen that a complex number takes the form $z = a + bi$. This form is called Cartesian form. When we are given a complex number in Cartesian form it is straightforward to plot it on an Argand diagram and then find its modulus and argument.

Instead of starting with the Cartesian form, sometimes the modulus, r say, and argument, θ say, are given to us. When this happens we are dealing with the polar form. In polar form we write

$$z = r,\theta$$

This means that z is the complex number with modulus r and argument θ.

Polar form:

$$z = r,\theta$$

Example 1

Plot the complex number $z = 4\angle 40^\circ$ on an Argand diagram and find its Cartesian form.

Solution. The Argand diagram in Figure 1 shows the complex number with modulus 4 and argument 40°.

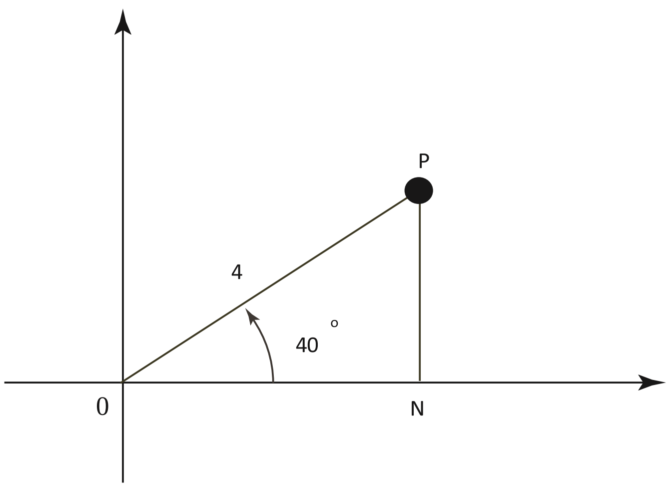


Figure 1. The complex number $z = 4\angle 40^\circ$.

We can use trigonometry to find the Cartesian form:

$$\cos 40° = \frac{ON}{4}$$

so that

$$ON = 4\cos 40° = 3.06$$

Similarly,

$$\sin 40° = \frac{NP}{4}$$

$$NP = 4sin40° = 2.57$$

$$NP = 4\sin 40° = 2.57$$

So the Cartesian form is $z = 3.06 + 2.57i$ (to 2 d.p.)

Having looked at a specific case, we will now look at the general case of a complex number $z = r\angle\,\theta$ as shown in Figure 2.

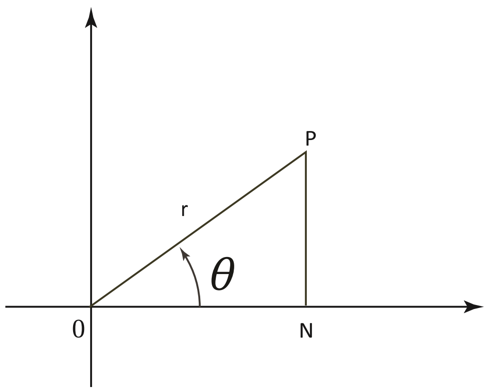


Figure 2. The complex number $z = r\angle\,\theta$.

To find the Cartesian form we need to find the length *ON* because this is the real part, and the length *NP* because this is the imaginary part.

$$\cos\theta = \frac{ON}{r} \quad \text{so that} \quad ON = r\cos\theta$$

$$\sin\theta = \frac{NP}{r} \quad \text{so that} \quad NP = r\sin\theta$$

We can now write down the Cartesian form: $z = r\cos\theta + i\,r\sin\theta$.

Equivalent forms of the complex number z:

$$z = r, \theta = r(\cos\theta + \mathrm{i}\sin\theta)$$

Example 2

Convert the complex number $z = 6\angle 110^{\circ}$ to Cartesian form.

Solution. The complex number is shown in Figure 3.

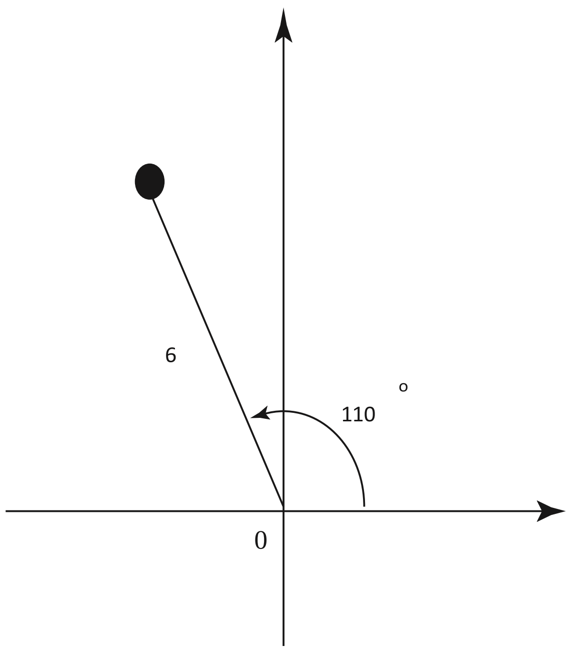


Figure 3. The complex number $z = 6\angle 110°$.

Using the previous result:

$$z = 6(cos110° + i\,sin110°)$$

Using a calculator we find

$$z = -2.05 + 5.64i \text{ (to 2 d.p.)}$$

Comparing this solution with the diagram in Figure 3 shows that, as expected from the diagram, the real part is negative.

Example 3

Express $4 + 3i$ in polar form.

Solution

The polar form $z = a + bi = r\,(cos\,\theta + i\,sin\,\theta)$

To find $r = |z| = \sqrt{4^2 + 3^2}$

$= \sqrt{25} = 5$

To find θ, $\tan\theta = \left(\frac{b}{a}\right)$

$= \frac{3}{4} = 0.75$

$\theta = 36.87^0$

r $(\cos\theta + i\sin\theta) = 5\,(\cos 36.87^0 + i\sin 36.87^0)$ to 2 d.p.

Example 4

z =a + b*i* is the rectangular form of a complex number whose polar representation is

$z = 5.831\,(cos\,59.04° + i\,sin\,59.04°)$. Determine the value of **a** and **b** hence re-write the rectangular form.

Solution

From the general equation of polar representation is

$z = \mathbf{r}(\cos\theta + i\sin\theta)$

$r = 5.831$ and $\theta = 59.04$

Using $\tan\theta = \left(\frac{b}{a}\right)$, $a = r\cos\theta$ and $b = r\sin\theta$

And from $r^2 = a^2 + b^2$ (Pythagoras theorem),

We obtain;

$5.831^2 = a^2 + b^2$

$34 = a^2 + b^2$ ……….(i)

From $a = r\cos\theta$, $a = 5.831\cos 59.04$

$= 3$

$a = 3$……….ii)

Replacing (ii) into (i), we get

$34 = 3^2 + b^2$

$34 = 9 + b^2$

$b^2 = 34 - 9 = 25$

$b = \sqrt{25} = 5$

Hence $\underline{\underline{z = 3 + 5i}}$

Exercise 6: Work in pairs.

1. let $z_1 = 3i$ and $z_2 = 2 - 2i$
   a) compute $|z_1 + z_2|$ and $|z_1 - z_2|$
   b) Express z_1 and z_2 in polar form
2. Express the following complex numbers to their polar form.
   i) $z = 8 + 6i$

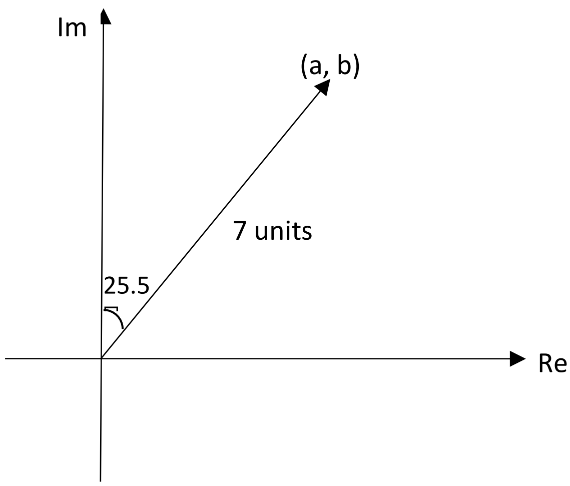
ii) $z = 3 - 4i$

iii) $z = \frac{(-0.5i - 4.9)}{7}$

3. Express the following complex numbers to rectangular form correct to 1 decimal point.

   a) $z = 5(\cos 45\square - 5isin\ 45°$
   b) $zz = 6(\cos 30° - i \sin 30°)$
   c) $z = 0.5\ (cos\ 35.6° + i\ sin\ 35.6°)$

4. The diagram below shows a point p(a, b) represented in a complex plane. determine its co-ordinates



Uses of complex numbers

Complex numbers can be used to solve quadratics for zeroes. The quadratic formula solves $ax^2 + bx + c = 0$ for the values of x. If the formula provides a negative in the square root, complex numbers can be used to simplify the zero.

Complex numbers are used in electronics and electromagnetism. A single complex number puts together two real quantities, making the numbers easier to work with. For example, in electronics, the state of a circuit element is defined by the voltage (V) and the current (I). Circuit elements can also have a capacitance (c) and inductance (L) that describes the circuit's tendency to resist changes in V and I. Rather than describing the circuit element's state by V and I, it can be described as $z = V + Ii$. The laws of electricity can then be expressed using the addition and multiplication of complex umbers.

As mentioned before, this can also be applied to electromagnetism. Instead of being described as electric field strength and magnetic field strength, you can create a complex number where the electric and magnetic components are the real and imaginary numbers.

Exercise 7: Work in groups.

1. Add and express in the form of a complex number $a + b\,i$.

$$\frac{(2 + 3i) + (-4 + 5i) - (9 - 3i)}{3}$$

2. Multiply and express in the form of a complex number $a + b\,i$.

$$(-5 + 3i)(-4 + 8i)$$

3. Divide and express in the form of a complex number a + b i.

$$(-1 - 2i) / (-4 + 3i)$$

4. Find the complex conjugate to.

$$1 + 8i$$

5. Express in the form of a complex number $a + b\,i$.

$$(-5 - i)(-7 + 8i) / (2 - 4i)$$

6. Express in the form of a complex number $a + b\,i$.

$$-(7 - i)(-4 - 2i)(2 - i)$$

7. Express in the form of a complex number $a + b\,i$.

$$i / (1 - i)$$

8. Solve for x and y where x and y are real numbers.

$$2y + ix = 4 + x - i$$

9. Find a and b, where a and b are real numbers so that

$$a + i\,b = (2 - i)2$$

10. Find the complex conjugate to $-3\,i$.

Exercise 8: Application Questions: Work in groups.

1. In an AC (alternating current) circuit, if two sections are connected in series and have the same current in each section, the voltage is given by $V = V_1 + V_2$. Find the total voltage in a given circuit if the voltages in the individual sections are $V_1 = 10.31 - 5.97i$ and $V_2 = 8.14 + 3.79i$.

2. The impedance Z in an AC (alternating current) circuit is a measure of how much the circuit impedes (hinders) the flow of current through it. The impedance is related to the voltage V and the current I by the formula$V = IZ$.
   If a circuit has a current of $(0.5 + 2.0i)$ amps and an impedance of $(0.4 - 3.0i)$ ohms, find the voltage.

UNIT 2

MEASUREMENT AND TRIGONOMETRY

What is area?

Area is the amount of space (on the inside) that a shape has. Imagine that you have a rectangle and a lot of one-inch squares cut out of construction paper. The number of one-inch squares it takes to completely cover the rectangle with no gaps or overlaps is the area of the rectangle. If it took 42 of those one-inch squares to cover the rectangle, that rectangle would have an area of 42 square inches. Area is always measured in square units, such as square inches, square centimeters, square feet, etc. It is also written as 42 sq. inches, or 42 in^2.

Area of Basic Shapes

To find the area of a shape, you could cover it with a lot of squares and then count, but that takes a lot of time. There is a quicker way. Formulas allow you to find the area of common shapes quickly. All you have to do is take a couple of measurements and do some computations. Many times the measurements will have been given to you, making it even easier. One of the simplest formulas is the area of a rectangle. The area of a rectangle (A) is length (l) multiplied by width (w). If you know the length and width of the rectangle, you multiply the two together to get the area. Other common shapes are often related so can be derived from one another, rather than memorizing lots formulae.

Area of Irregular Shapes

Many shapes are not the basic shapes (rectangles, triangles, circles, etc.) that we have formulas for. There are times when you need to find the area of a shape that is not a regular shape. One method of finding the area of an irregular shape is to divide the shape into smaller shapes which you do have the formula for. Find the area of all of the smaller shapes and then add your areas together.

At times, you are finding the area of an irregular shape that appears to be a shape inside of another shape. In this case, you are taking away some area. You would subtract the excess area of the inside the outside shape from the area of the outside shape.

How to use a grid paper to approximate an irregular area

Trace your hand on a piece of paper. Think about how you might determine the area of your handprint.

Activity 1

Will the amount of area covered differ if you trace your hand with your fingers close together or spread apart? Explain.

Units for measuring area must have the following properties:

- The unit itself must be the interior of a simple closed shape.
- The unit, when repeated, must completely cover the object of interest, with no holes or gaps (like a tessellation). Many polygons (e.g., rectangles, rhombuses, and trapezoids) and irregular shapes (e.g., L shapes) have this property and can thus be used as units of measurement.

Activity 2

1. What units might you use to determine the area of your handprint?
2. Why is a small circle not suitable as the unit of measurement?

Activity 3

1. One method for finding the area of an irregular shape is to count unit squares. Use centimeter grid paper to determine the area of your handprint. What are the disadvantages of this method?
2. Another method is to subdivide your handprint into sections for which you can easily calculate the area. Find the area of your handprint using this method. Does using the two methods result in the same area?

Up until now, you have been approximating the area of your handprint. In other words, your measurements were not exact.

Activity 4

What can you do to make your approximation more accurate? Explain why this approach will lead to a better approximation. Another way to approximate the area of a handprint or any other irregular shape is to determine the number of squares that are completely covered and the number of squares that are partially covered. Average these two numbers to get an approximate area in the number of square units.

Activity 5

Think about the following statement: If you repeatedly use a smaller and smaller unit to calculate the area of an irregular shape, you will get a closer and closer approximation and eventually find the exact area. What do you think of this line of reasoning? Explain.

Activity 6

The palm of your hand is about one percent of your body's surface area. Doctors sometimes use this piece of information to estimate the percent of the body that is affected in burn victims. Use your data to approximate the amount of skin on your body.

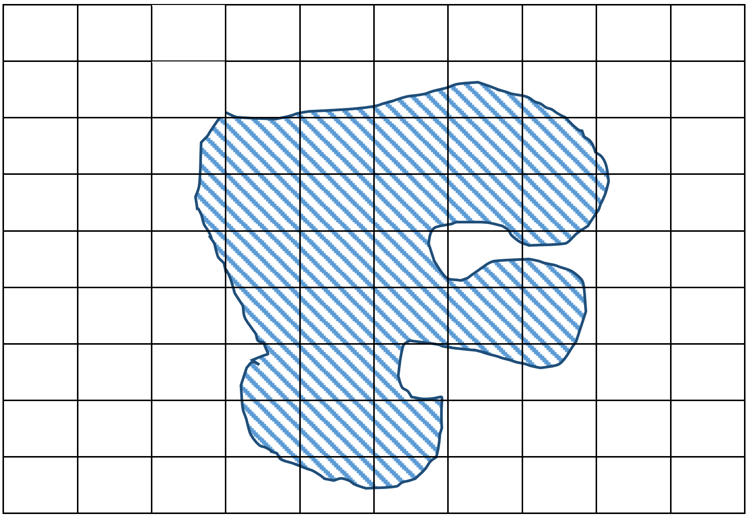
Activity 7: Work in pairs

Given the irregular shape;

1. Sub-divide the area into squares of a unit length.
2. Mark all the whole squares within the area.
3. Mark all incomplete squares with a different mark (e.g. xx).
4. Count the complete squares.
5. Count the incomplete squares.
6. The area is therefore approximated by;
   i. Area = number of complete squares + $\frac{1}{2}$ (number of incomplete squares)

Example

1. The map of Imatong forest is as shown in the figure below. The scale of the map is 1:100000. Determine the area;
   (i) In square centimeters.
   (ii) In hectares.



Solution

(i) Number of complete squares =

(ii) Number of incomplete squares =

Area = number of complete squares + $\frac{1}{2}$ (number of incomplete squares)

If 1cm rep 100,000 cm on the map

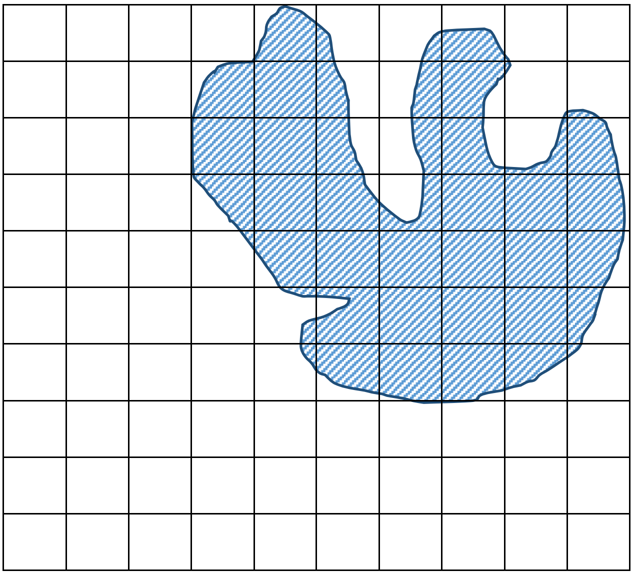
$1cm^2$ ⟹ $(100,000 \times 100,000)cm^2$

$10,000,000,000cm^2$

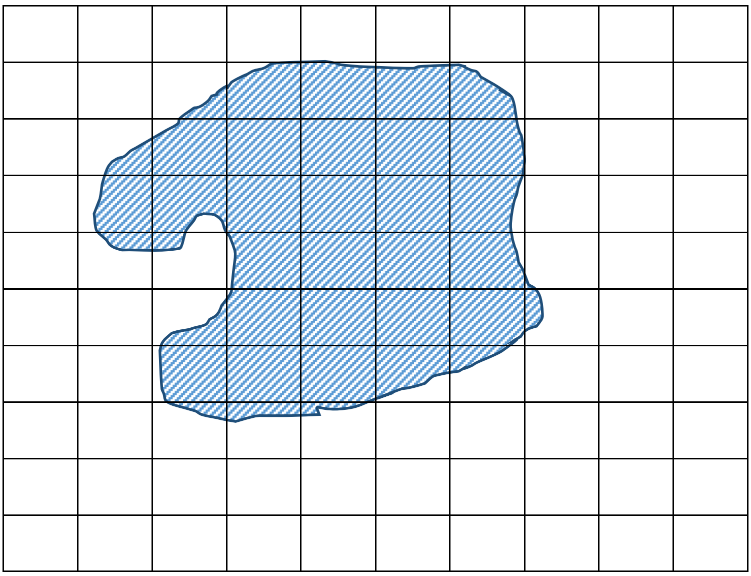
$10\ ha$

Exercise 1

1. Estimate the area of shaded region in the figure below in the square units.



2. The figure on the following page shows a map a given country. If the scale is 1:200,000.



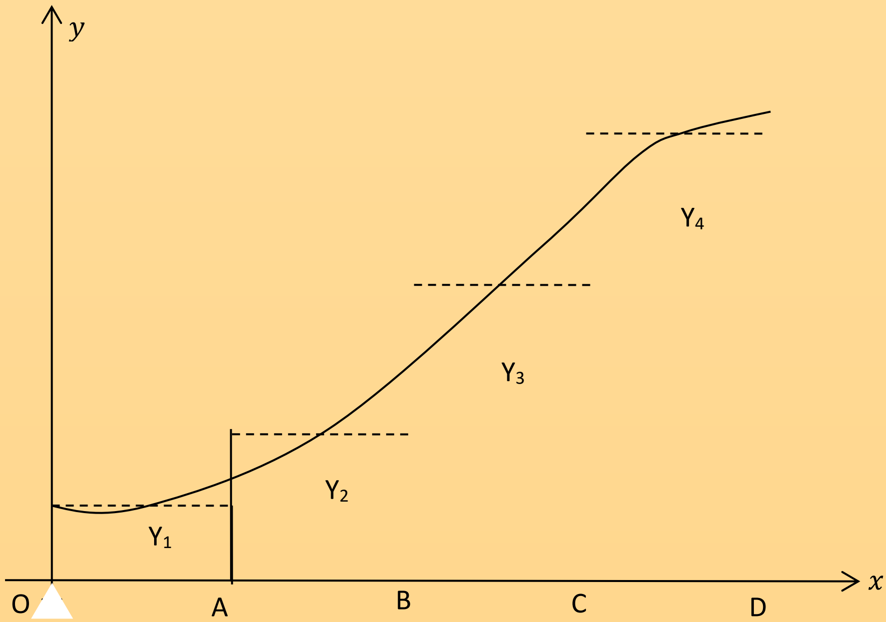
Determine;

(i) The area in square centimeters.

(ii) The area in hectares.

Area estimation by using mid- ordinate rule

This involves dividing an irregular shape into rectangular strips all of equal width h, as follows.



(i) Divide the base of the irregular area into rectangular strips each of equal width, *h*

(ii) Get the mid-point of OA, AB, BC, CD. Draw a straight vertical line from the mid-point to the curve.

Call these lines mid-ordinates (y_1, y_2, y_3 and y_4).

The area of the irregular shape is equivalent to the sum of the areas of all the rectangles whose lengths are y_1, y_2, y_3, y_4 and a constant width h.

This given by;

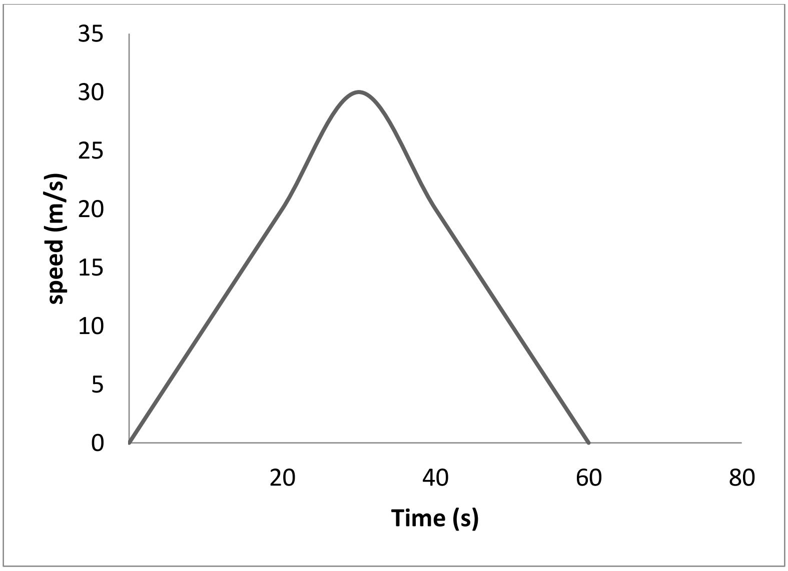
$$A = (h \times y1_1) + (h \times y_2) + (h \times y_3) + (h \times y_4)$$
$$= h\,(y_1 + y_2 + y_3 + y_4)$$

Area general formula, given n rectangles.

$$A = h\,(y_1 + \cdots . y_n) = \ interval\ width \times \ sum\ of\ mid\ ordinates$$

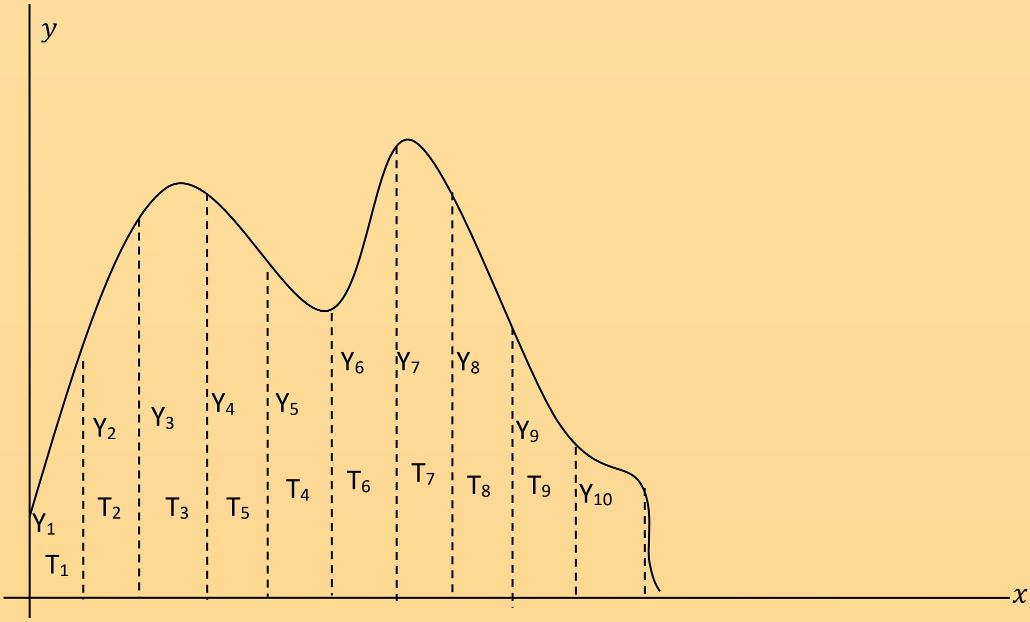
Exercise 2

1. Estimate the distance covered by the particle whose speed- time graph is shown below, using the mid-ordinate rule.



Area estimation using a trapezium rule.

With the trapezium rule, instead of approximating area by using rectangles you approximate area with trapeziums.



Area of trapezium 1, $T_1 = \frac{1}{2} h(y_1 + y_2)$

$$T_2 = \frac{1}{2} h(y_2 + y_3)$$

$$T_3 = \frac{1}{2} h(y_3 + y_4) \ldots$$

$$T_9 = \frac{1}{2} h(y_9 + y_{10})$$

→ Since all the heights appear twice in the formulae above except y_1 and y_{10} , the formulae can be simplified as follows.

$$\frac{1}{2} h(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + 2y_8 + 2y_9 + 2y_{10})$$

$$\frac{1}{2} h\,(y_1 + y_{10} + 2(y_2 \ldots\ldots + y_{10})$$

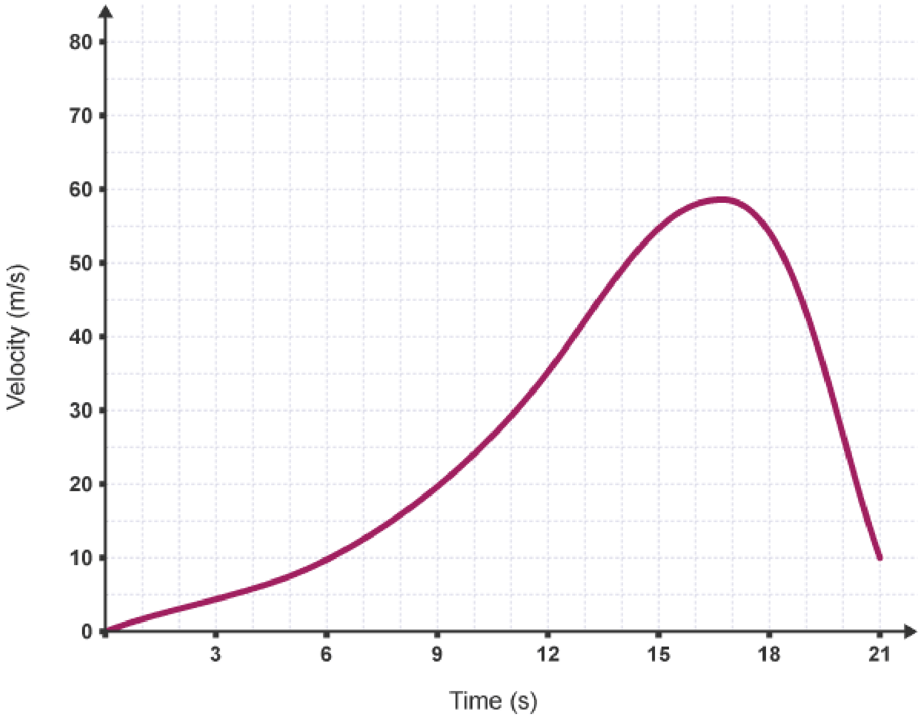
Therefore, for a number $\boldsymbol{n}$ of trapezia,

The area $A = \frac{1}{2} h\{(y_1 + y_n) + 2(y_2 + y_3 \ldots \ldots y_{n-1})\}$

For better estimation of area, the h, needs to be smaller i.e. the smaller the h, the more accurate the answer.

Example 1

The following graph shows a journey made by a motorcyclist on returning home from the nearest bike shop.



Use the trapezium rule with five strips to approximate the distance travelled between $t = 3$ and $t = 18$ seconds.

The distance needed is between $t = 3$ and $t = 18$, and the corresponding y-coordinates would be:

$t = 3,\ y_{first} = 5$

$t = 6,\ y_2 = 10$

$t = 9,\ y_3 = 20$

$t = 12,\ y_4 = 35$

$t = 15,\ y_5 = 55$

$t = 18,\ y_{last} = 55$

$A = \frac{1}{2} \times h(y_{first} + y_{last} + 2(Sum\ of\ the\ rest))$

$A = \frac{1}{2} \times 3 \times (5 + 55 + 2(10 + 20 + 35 + 55))$

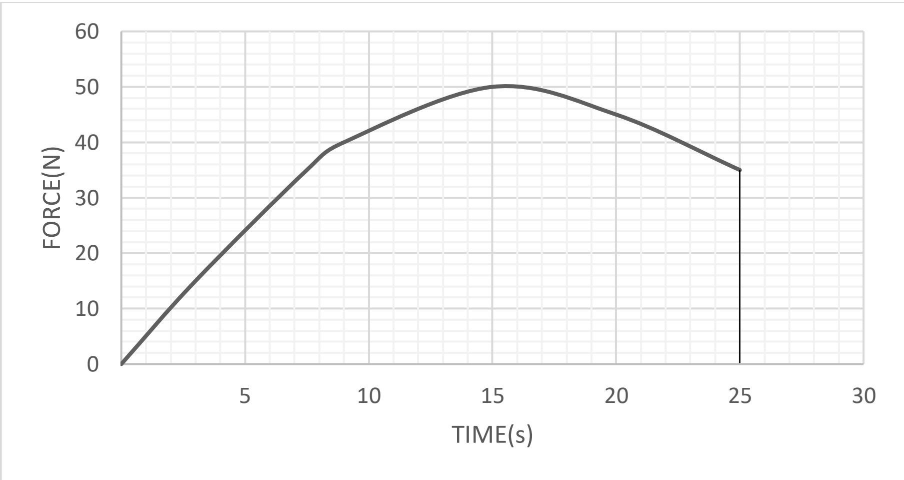
$A = 450\ m$

Exercise 3

1. A soldier projected a bullet and recorded the speed after every 2 second as shown in the table below. If the bullet hit a given wall, use the trapezoidal rule to estimate how far the soldier was from the wall.

Time (s)	0	2	4	6	8	10	12	14
Speed (m/s)	0	2.4	5.0	4.8	6.4	6.2	4.6	2.8

2. In attempt to determine the change in momentum of a given body. John recorded the force acting on the body at different time intervals and hence drew the graph below.



Given that the change in momentum of the body is given by the area under the graph from $t = 0$ to $t = 25$ seconds, estimate the change in momentum using trapezium rule using 5 trapezia.

Task

Use the equation $y = 3x^2 - 2x + 4$ to complete the table below.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y			5					33.75	

a) Use integration to find the exact the area bounded by *x*- axis, *y*- axis and the line $x = 4$

b) Use the mid-ordinate rule to estimate the area bounded by *x*- axis, *y*- axis and the line $x = 4$ with 8 rectangles.

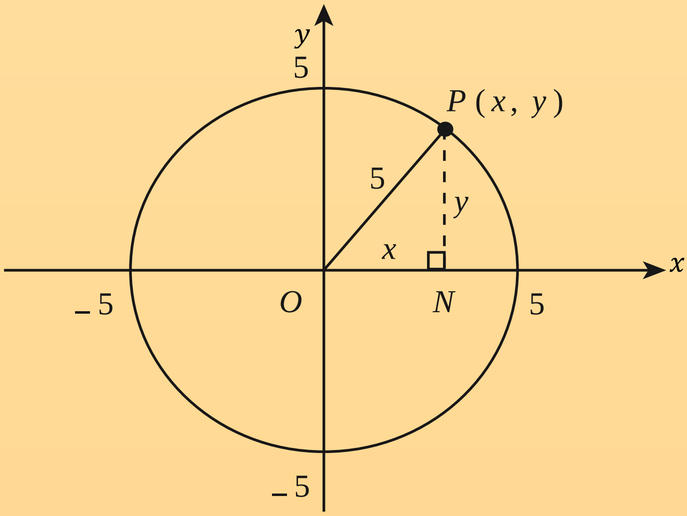
c) Use trapezium rule to estimate the area bounded by *x*- axis, *y*- axis and the line $x = 4$ with 8 trapezia.

d) Comment on your different results.

Equation of a circle

The equation of a circle centre the origin

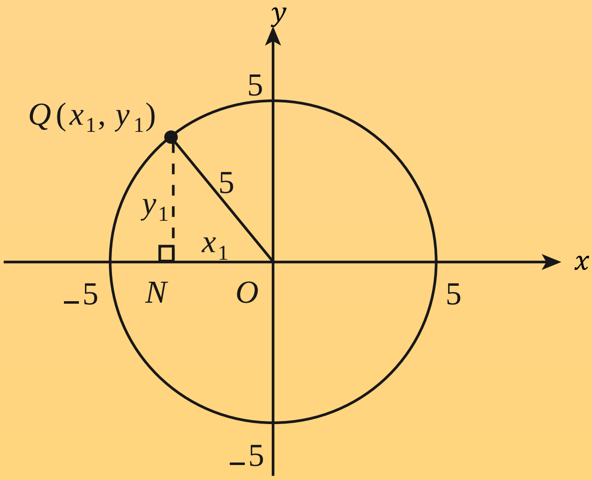
The simplest case is that of a circle whose centre is at the origin. Let us take an example. What will be the equation centred on the origin with radius 5 units?



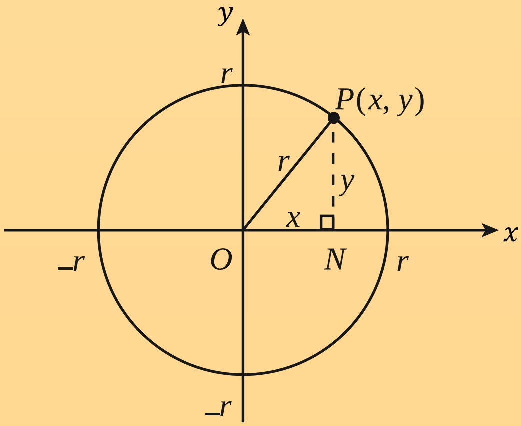
If we take any point $P(x, y)$ on the circle, then $OP = 5$ is the radius of the circle. But OP is also the hypotenuse of the right-angled triangle OPN, formed when we drop a perpendicular from P to the x-axis. Now in the right-angled triangle, $ON = x$ and $NP = y$. Thus, using the theorem of Pythagoras,

$$x^2 + y^2 = 5^2 = 25.$$

And this equation is true for any point on the circle. For instance, we could take a point $Q(x_1,y_1)$ in a different quadrant.



Once again, we can drop a perpendicular from Q to the x-axis. And now we can use the right-angled triangle OQN to see that $x_1^2 + y_1^2 = 5^2$. So the co-ordinates (x_1,y_1) of the point Q also satisfy the equation $x^2 + y^2 = 25$.
We shall now take the radius of the circle to be r.



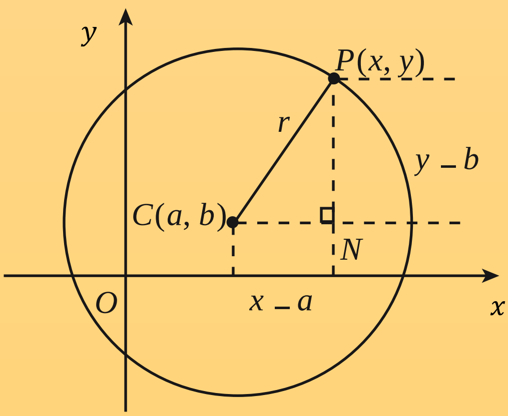
If we take any point $P(x, y)$ on the circle, then $OP = r$ is the radius of the circle. But OP is also the hypotenuse of the right-angled triangle OPN, formed when we drop a perpendicular from P to the x-axis. In the right-angled triangle, $ON = x$ and $NP = y$. Thus, using the theorem of Pythagoras,

$$x^2 + y^2 = r^2,$$

and this is the equation of a circle of radius r whose centre is the origin $O(0,0)$.

The general equation of a circle

What is the equation of a circle of radius r, centred at the point $C(a, b)$?



We shall take a horizontal line through the centre C and drop a perpendicular from P to meet this horizontal line at N. Then again we have a right-angled triangle CPN, where $CP = r$ is the hypotenuse, and where we have $CN = x - a$ and $PN = y - b$. Thus using Pythagoras again we have

$$\overrightarrow{CN}^2 + \overrightarrow{PN}^2 = \overrightarrow{CP}^2,$$

so that

$$(x - a)^2 + (y - b)^2 = r^2.$$

Expanding the brackets gives

$$x^2 \;\; 2ax + a^2 + y^2$$
$$-2by + b^2 = r^2,$$

and if we bring r^2 to the left-hand side and rearrange we get

$$x^2 - 2ax + y^2 - 2by + a^2 + b^2 -$$

$r^2 = 0$. It is a convention, at this point, to replace a by g and b by f. This gives

$$x^2 + 2gx + y^2 + 2fy + g^2 + f^2 - r^2 = 0.$$

Now look at the last three terms on the left-hand side, $g^2 + f^2 - r^2$. These do not involve. Substituting this into x or y at all, so together they just represent a single number that we can call c the equation finally gives us

$$x^2 + 2gx + y^2 + 2fy + c = 0.$$

This is the general equation of a circle. We can recognise it, because it is quadratic in both x and y, and it has two additional properties. First, there is no term in xy. And secondly, the coefficient of x^2 is the same as the coefficient of y^2. (In fact, our equation has both coefficients equal to 1, but you can always multiply an equation by a non-zero constant to obtain another valid equation, and so we must allow for this possibility.) The centre of the circle is then at $(a,b) = (-g,-f)$ and, since $c = g^2 + f^2 - r^2$, we have

$$r^2 = g^2 + f^2 - c,$$

so that the radius of the circle is given by

$$r = \sqrt{g^2 + f^2 - c}.$$

Example 1

Find the centre and radius of the circle,

$$2x^2 + 2y^2 - 8x - 7y = 0.$$

Solution

Notice that this is the equation of a circle, even though the coefficients of x^2 and of y^2 are not equal to 1. But we can divide throughout by 2, and we get

$$x^2 + y^2 - 4x - \tfrac{7}{2}y = 0.$$

If we compare this with the standard equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

we see that $g = -2$ and, $f = -\frac{7}{4}$ so the centre of $(-g, -f) = (2, \frac{7}{4})$ the circle is. We also see that $c = 0$, so we find the radius by calculating

$$\begin{aligned} r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(-2)^2 + \left(-\tfrac{7}{4}\right)^2} \\ &= \sqrt{4 + \tfrac{49}{16}} \\ &= \sqrt{\tfrac{113}{16}} \\ &= \tfrac{1}{4}\sqrt{113}. \end{aligned}$$

Alternatively, we could try completing the square to regain the form $(x - a)^2 + (y - b)^2 = r^2$. So we start with our equation

$$2x^2 + 2y^2 - 8x - 7y = 0,$$

and again we divide by 2 to get

$$x^2 + y^2 - 4x - \tfrac{7}{2}y = 0.$$

Collecting the x terms together and the y terms together, we get

$$x^2 - 4x + y^2 - \tfrac{7}{2}y = 0,$$

and then completing the square gives us

$$(x-2)^2 - 4 + \left(y - \frac{7}{4}\right)^2 - \frac{49}{16} = 0$$

so that

$$(x-2)^2 + \left(y - \frac{7}{4}\right)^2 = \frac{113}{16}.$$

We can now see that the centre of the circle is $(2, \frac{7}{4})$ and the radius is $\frac{1}{4}\sqrt{113}$.

Example 2

Find the equation of a circle Centre (-2, 1) and radius 3 units

Solution

Equation of a circle Centre (a, b) and radius r units is given by $(x-a)^2 + (y-b)^2 = r^2$

With Centre (-2, 1) and radius 3 units, substitute the values to get.

$$(x+2)^2 + (y-1)^2 = 3^2$$

By expansion

$$x^2 + 2x + 4 + y^2 - 2y + 1 = 9$$

$$x^2 + 2x + y^2 - 2y + 5 = 9$$

$$x^2 + 2x + y^2 - 2y - 4 = 0$$

Exercise 4

1. Find the equation of a circle given that the coordinate's center and one point on the circumference.
   a) Centre (0,0) and passing through (2, -1)

b) Centre (2,1) and passing through (-3,2)

c) Centre (4,1) and passing through (0,-3)

d) Centre (4,5) and passing through (7,6)

e) Centre (1,-2) and passing through (3,3)

2. Show that the equation of the circle is given whose Centre is (-3,1) and radius 4 units is given by:

$$x^2 + 6x + y^2 - 2y = 6$$

2. Find the Centre and the radius of each of the following circles:

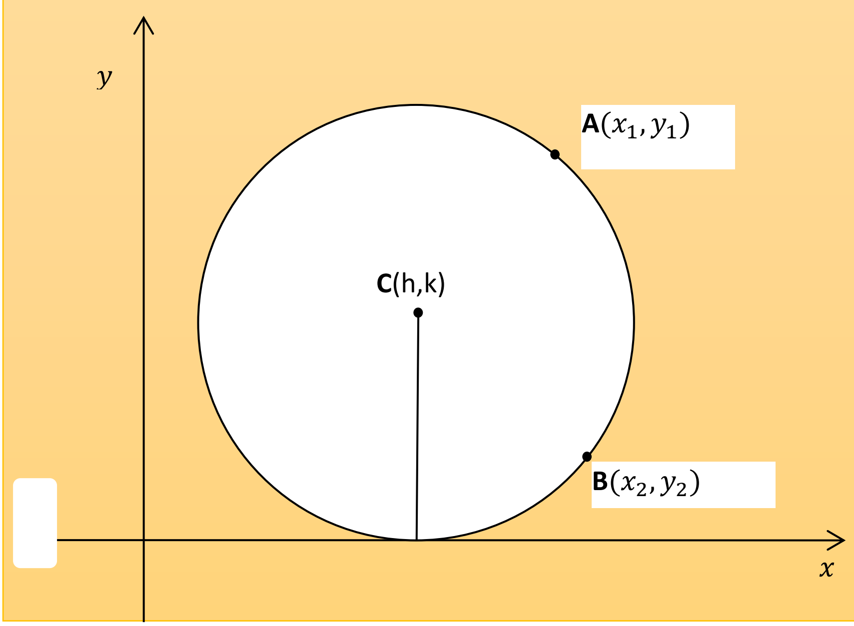
a) $(x-2)^2 + (y-2)^2 = 25$

b) $(x+1)^2 + y^2 = 16$

c) $x^2 + y^2 = 49$

d) $x^2 - 4x + y^2 + 6y - 4 = 0$

The equation of a circle passing through two points and touching the x-axis



In the circle above, k which is the y coordinate of the Centre is equal to the radius r**.** **A** and **B** are the two given points.

The equation of the above circle would therefore be:

$(x-h)^2 \ + \ (y-k)^2 \ = \ r^2$

$(x-h)^2 + (y-k)^2 = r^2$

But since $k = r$ then

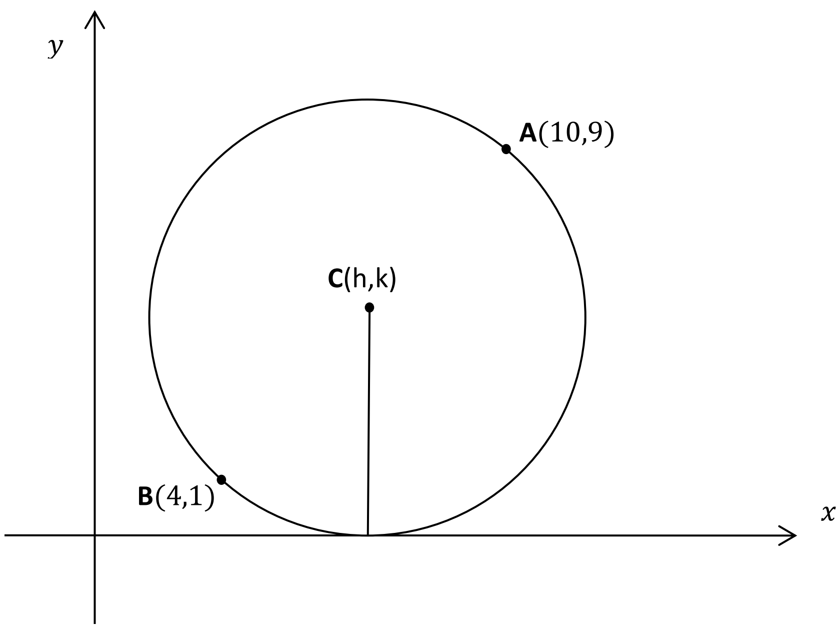
$(x-h)^2 + (y-k)^2 = k^2$

Example

Find the equation of the circle passing through (4,1) and (10,9) and touching the x-axis.

Solution

Sketch



With Centre (h, k) the equation the circle is

$(x-h)^2 + (y-k)^2 = r^2$

But since $k = r$

$$(x-h)^2 + (y-k)^2 = k^2$$

By expanding the equation above

$$x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = k^2$$

$$x^2 - 2xh + h^2 + y^2 - 2yk = k^2 - k^2$$

$$x^2 - 2xh + h^2 + y^2 - 2yk = 0$$

Substitute each of the 2 given points for (x, y) substituting (4,1)= (x, y)

$$4^2 - 2(4)\,h + h^2 + 1^2 - 2(1)k = 0$$

$$16 - 8h + h^2 + 1 - 2k = 0$$

$17 - 8h + h^2 - 2k = 0$…………………. Equation (i)

Substituting (10,9) $= (x, y)$

$$10^2 - 2(10)h + h^2 + 9^2 - 2(9)k = 0$$

$$100 - 20h + h^2 + 81 - 18k = 0$$

$181 - 20h + h^2 - 18k = 0$ … … … … … …equation (ii)

By taking equation (i) and (ii) eliminate k

By multiplying equation (i) by 9 we get equation (iii)

$$\{17 - 8h + h^2 - 2k = 0\} \times 9$$

$$153 - 72h + 9h^2 - 18k = 0$$

Subtract equation (iv) from (iii)

$$153 - 72h + 9h^2 - 18k = 0 \text{ ………….. (iii)}$$

$$\underline{181 - 20h + h^2 - 18k = 0 \text{ … … … … … …. (iv)}}$$

$$-28 - 52h + 8h^2 + 0 = 0$$

$$8h^2 - 52h - 28 = 0$$ (divide both sides by 4)

$$2h^2 - 13h - 7 = (2h + 1)(h - 7) = 0$$

Substitute the two values of *h* in the either equation (i) and (ii) to get the value of *k*.

When *h= 7*

$$17 - 8(7) + (7)^2 + 2k = 0$$

$$17 - 56 + 49 = 0$$

2k=10

k=5

Centre (h, k) = (7, 5)

When *h= -0.5*

$$17 - 8(-0.5) + (-0.5)^2 - 2k = 0$$

$$17 + 4 + 0.25 = 2k$$

$$k = 10.625$$

Centre (h, k) = (-0.5, 10.625)

With Centre (7, 5) the equation of the circle is,

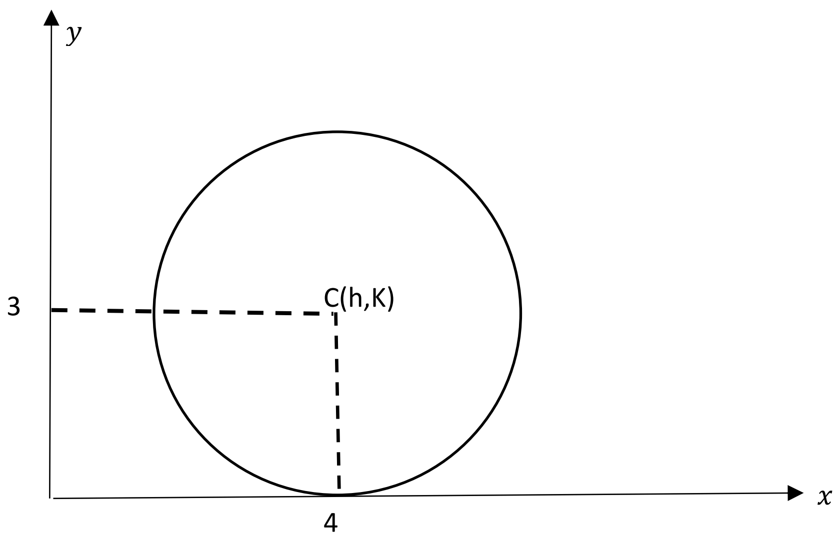
$(x-7)^2 + (y-5)^2 = 25$

With Centre (-0.5, 10.625) the equation of the circle is,

$(x+0.5)^2 + (y-10.625)^2 = 112.89$ (to 2 d.p.)

Exercise 5

1. Find the equation of each of the following circles. Each of the circles touch the x-axis and passes through the set of points given.
    a) (2, 1) and (10, 7)
    b) (4, 2) and (11, 9)
    c) (6, 1) and (10, 11)
    d) (4, 0) and (9, 9)
2. The figure below shows a circle Centre *(h, k)* and touching the x-axis at x=4.



i. Find the values of h and k hence state the co-ordinates of the Centre of the circle.

ii. State the value of the radius of the circle.

iii. Find the equation of the circle.

Graphs of trigonometric functions

Plotting graphs of simple trigonometric functions

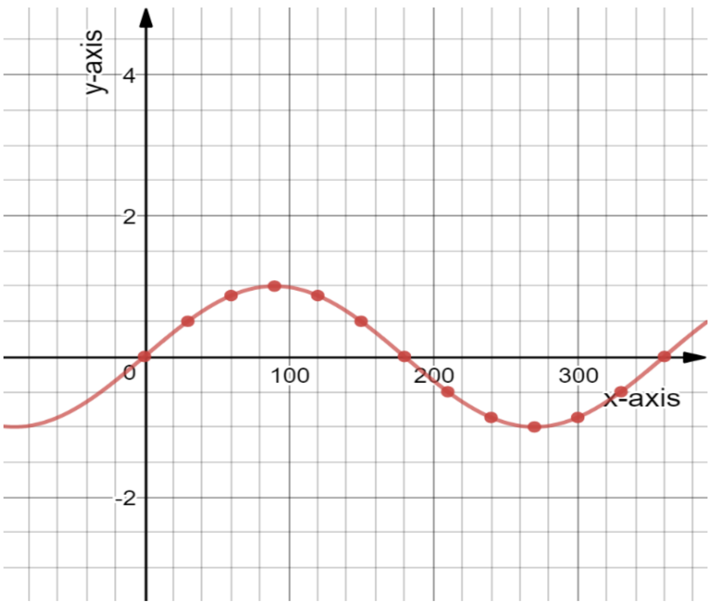
Simple trigonometric graphs such as $y = sin\ x$, $y = cos\ x$ and $y = tan\ x$ for $0° \leq x \leq 360°$ are easily drawn by choosing values of x that fall within the given ranges obtaining the corresponding values of y and then plotting them using a suitable scale.

Examples

i. $y = sin\ x.$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

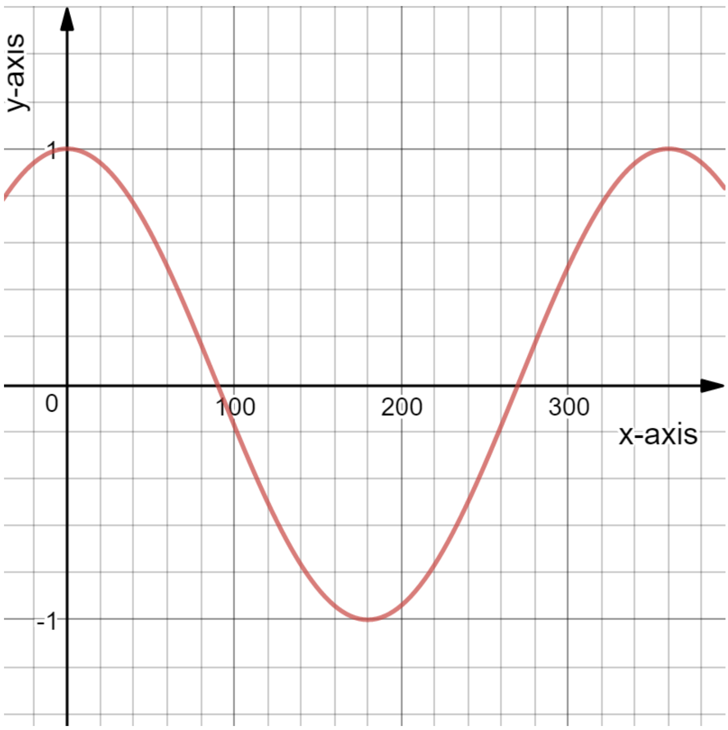
These are then plotted often choosing a convenient scale e.g.



ii. $y = \cos x$

x	0°	30°	60°	90	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

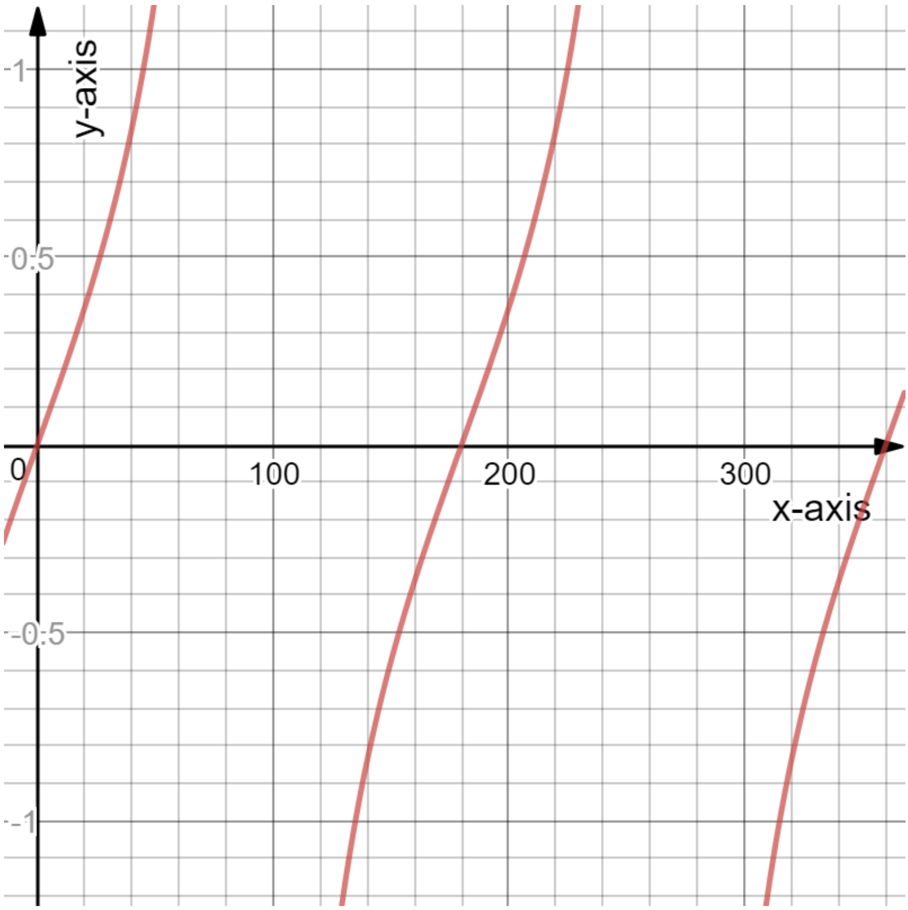
Using scales of 1cm rep $30^0$for x axis and 4 cm rep 1-unit y axis



iii. $y = \tan x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	0	0.58	1.73	∞	−1.73	−0.58	0	0.58	1.73	∞	−1.73	−0.58	0

The values of y for x = 90^0 and x = 270^0 are too large (undefined)



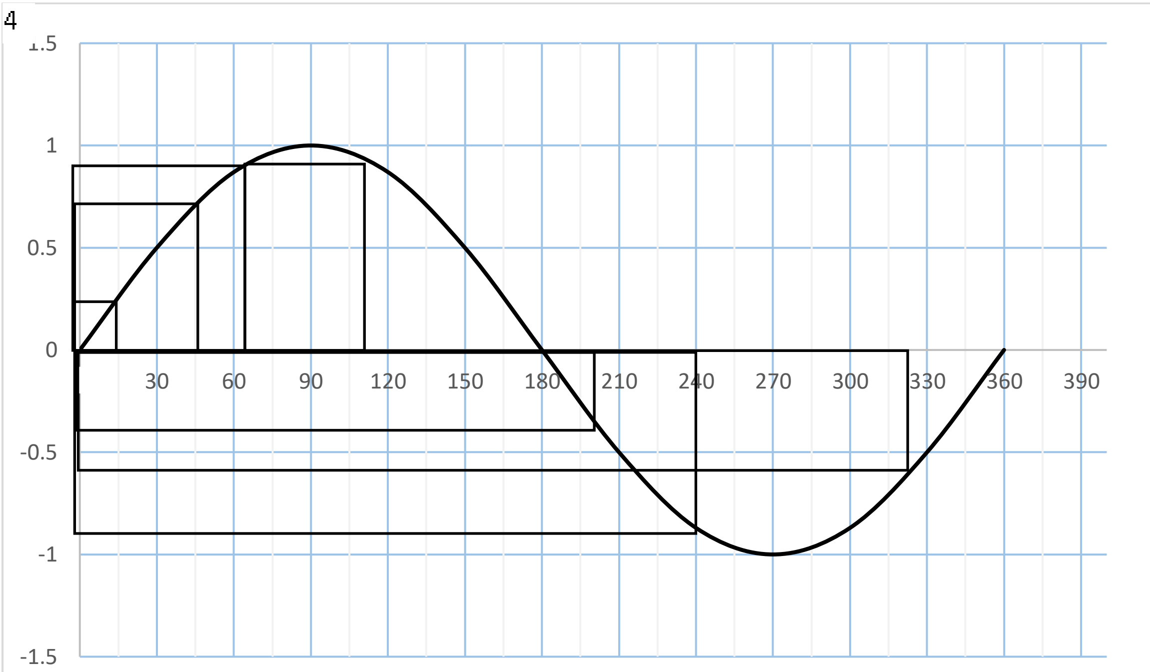
iv. Draw the graph of $y = sin\ x$ and use it to solve the following equations:

$sin\ x = 0.7, sin\ x = 0.3, sin\ x = -0.4$

Also use your graph to find sine of the following angles

i. 72^0
ii. 108^0
iii. 246^0
iv. 324^0

Solution



For $sin\ x = 0.7$

$x = 42°$

$sin\ x = 0.3$

$x = 18°$

$Sin\ x = -0.4$

$x = 202°$

$sin\ 246° = -0.9$

$sin\ 72° = 0.95$

$sin\ 108° = 0.95$

$sin\ 324° = -0.58$

Exercise 6

1. Draw the graph of trigonometric equation $y = \cos x$ for $0° \leq x \leq 360°$ and use it to solve the following equations.
   a) $Cos\, x = 0.5$
   b) $Cos\, x = -0.6$
   c) $Cos\, x = 0.72$
   d) $Cos\, x = -0.9$
2. Use the same graph to find the cosine of the following angles.
   a) 24^0
   b) 56^0
   c) 112^0
   d) 260^0
   e) 294^0
   f) 318^0
3. For the range of the values of x is $0° \leq x \leq 360°$, draw the following graphs.
   a) $y = 2\cos x$
   b) $y = \sin 2x$
   c) $y = \cos x + \sin x$

UNIT 3

INEQUALITIES AND VECTORS

Linear Inequalities

Inequality Signs

1. Greater than (>)
   Example: x is greater than 8 is written as $x > 8$.
2. Less than (<)
   Example: x is less than 10 is written as $x < 10$.
3. Greater than or equal to (≥)
   Example: m is greater than or equal to -2 is written as $m \geq -2$.
4. Less than or equal to (≤)
   Example: m is less than or equal to 3 is written as $m \leq 3$.

Formation of inequalities

Example 1

Aban wants to buy some shirts and some trousers. He buys at least 8 shirts but not more than 30 items altogether. The number of shirts bought should be more than twice the number of trousers bought, form three inequalities to represent the information above. Let x represent the number of shirts and y to represent the number of trousers.

$x \geq 8$ i.e. he can buy 8 or more shirts.

$x + y \leq 30$ i.e. the total number of both shirts and trousers should be equal to 30 or less.

$x > 2y$ Shirts should be more than twice trousers.

Example 2

A room is search that its width is 5 meters less than its length. The length is greater than 6m while the perimeter of the room is not greater than 40m. Given that the width of the room is **b** meters, find the range of values of **b** satisfying all the inequalities representing the given situation.

Solution

Width= $\boldsymbol{b}$

Length= b+5m

$\boldsymbol{b} + 5 > 6$ i.e. length is greater than 6.

$\boldsymbol{b} > 1$ ……... (i)

Perimeter $2(\boldsymbol{b} + \boldsymbol{b} + 5)$

$2(2\boldsymbol{b} + 5) = 4\boldsymbol{b} + 10$

$4\boldsymbol{b} + 10 \leq 40$ Perimeter is not greater than 40.

$4\boldsymbol{b} \leq 30$

$\boldsymbol{b} \leq 7.5$ ……... (ii)

From (i) and (ii), $1 < b \leq 7.5$.

Solution of Linear Programming by Graphs

Incase several inequalities in 2 variables are satisfied simultaneously, a region will be defined within the plane.

The line representing the corresponding equation in each inequality is drawn and the unwanted region is shaded.

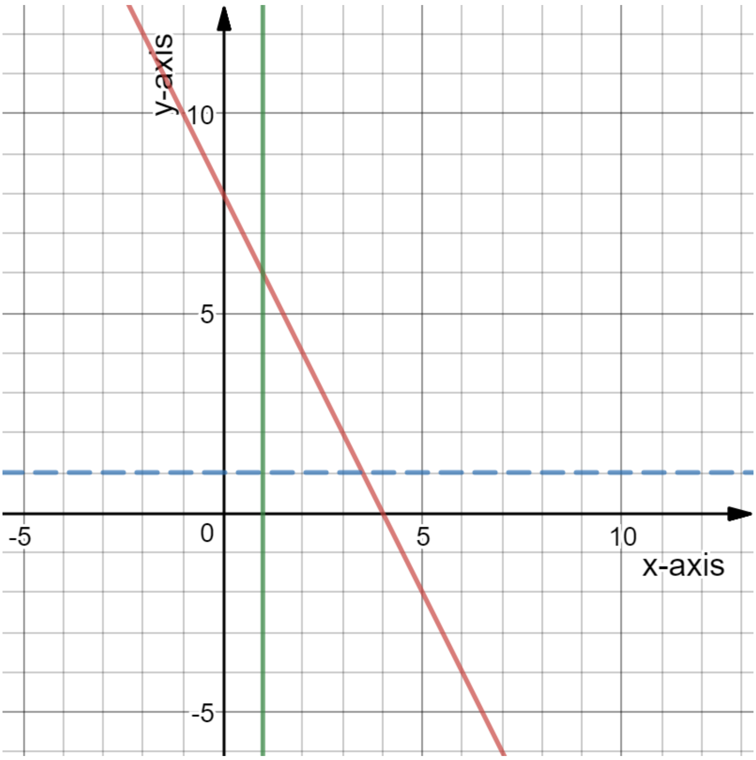
The points in the region that remains unshaded provides the solution of the inequalities.

Example 1

Show the region defined by the following inequalities.

$2x + y \leq 8, y \geq 1$ and $x > 2$.

From the inequalities above, draw the lines whose equations are; $2x + y = 8$, $y = 1$, $x = 1$



The region R defines the inequalities.

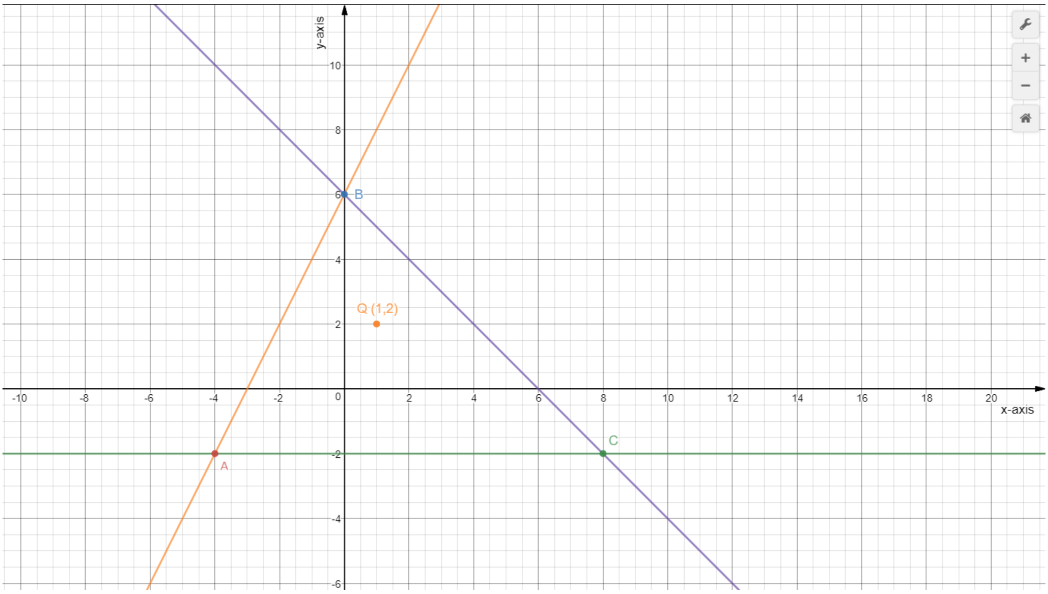
NB. The line $y = 1$ is a broken line.

The line representing the inequalities $>$ or $<$ should be broken lines while those representing $\geq$ or $\leq$ should be continuous lines.

Example 2

Find the inequalities that define the unshaded region R in the diagram below.

Solution.



First find the equation of each of the lines that bound region R i.e. line AB, BC, and AC.

The equation of line AB is $y = 2x + 6$.

The equation of line BC is $y = -x + 6$.

The equation of line AC is $y = -2$.

Taking a point Q within the region R as a reference point, we can obtain the 3 inequalities as follows:

Line AB; $y = 2x + 6$. Substituting the values of x and y at point Q in the equation will give us the inequality $y \leq 2x + 6$ since from the equation 2 < 2(1) + 6.

Line BC; $y = -x + 6$. Substituting the values of x and y in the equation with x *and* y co-ordinates of point Q gives us 2< (-1) + 6 therefore the inequality would be $y \leq -x + 6$ which can also be written as $y + x \leq 6$ or $y \leq 6 - x$

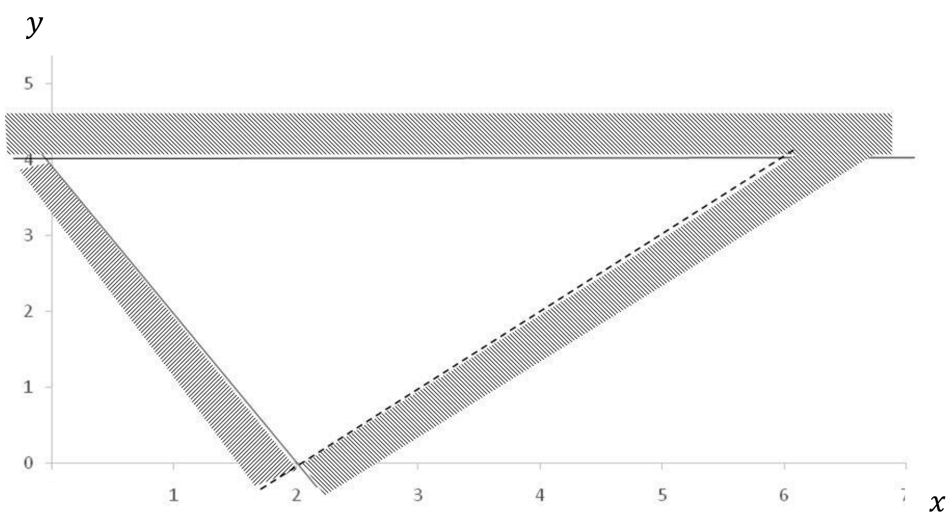
Line AC; $y = -2$ will give the inequality $y \geq -2$ which satisfies the region R.

The three inequalities therefore are

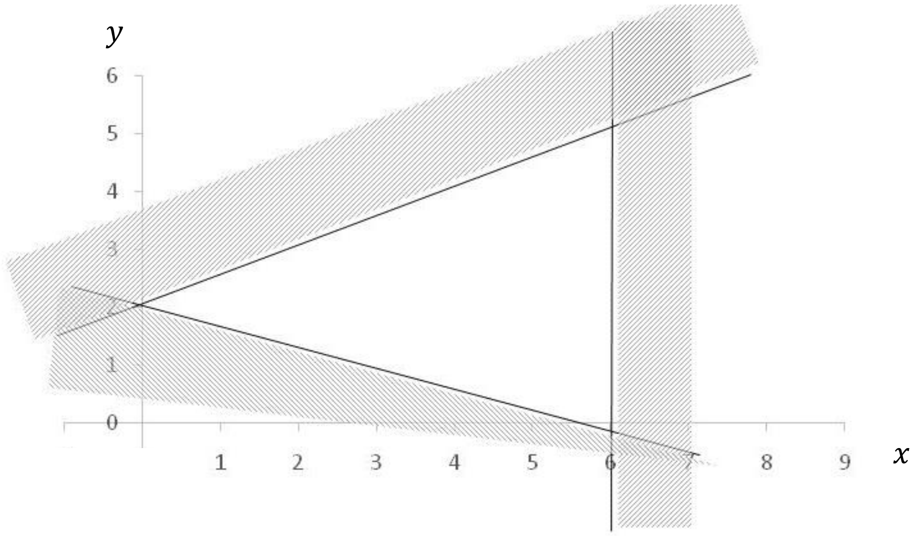
$y \leq 2x + 6.$
$y + x \leq 6.$
$y \geq -2.$

Exercise 1

1. Draw the graphs of the following inequalities. In each case show the region R that satisfies the inequalities.
   a) $x \geq 0, x + y \leq 3, y > x + 1.$
   b) $x \leq 3, x + 2y \geq 4, 4y - 3 \leq 8.$
   c) $y < x, x \leq 4, x + y < 6.$
2. In each of the following graphs, find the set of inequalities that represent region R.
   (a)



(b)



Optimization

There are real life situations that require one to work or make a choice from certain limits. We can form inequalities representing those situations and use **linear programming** to obtain the maximum and minimum values of these inequalities. This process is known as **optimization.**

Example

A farmer has 20 hectares of land on which he can grow maize and beans only. In a year he grows maize on more land than beans. It costs him SSP 4000 to grow maize per hectare and SSP 10 000 to grow beans per hectare. He is prepared to spend at most SSP 90 000 per year to grow the crops. He makes a profit of SSP 2000 from one hectare of maize and SSP 3000 from one hectare of beans. Find the maximum profit he can make from the crops in a year.

Solution

First formulate all the inequalities that represent the information above.

Let the number of hectares of maize= x and the number of hectares of beans= y.

Inequalities;

$x + y \leq 20.$

$x > y.$

$4\,000x + 10\,000y \leq 90\,000.$

$4x + 10y \leq 90.$

$x > 0.$

$y > 0.$

Draw the graphs representing the inequalities above.

i) For

$x + y \leq 20$

$x + y = 20$

x	0	5	10	15	20
y	20	15	10	5	0

ii) For

$4x + 10y \leq 90$

$4x + 10y = 90$

$y = (90 - 4x)/10$

x	0	5	10	15
y	9	7	5	3

iii) For

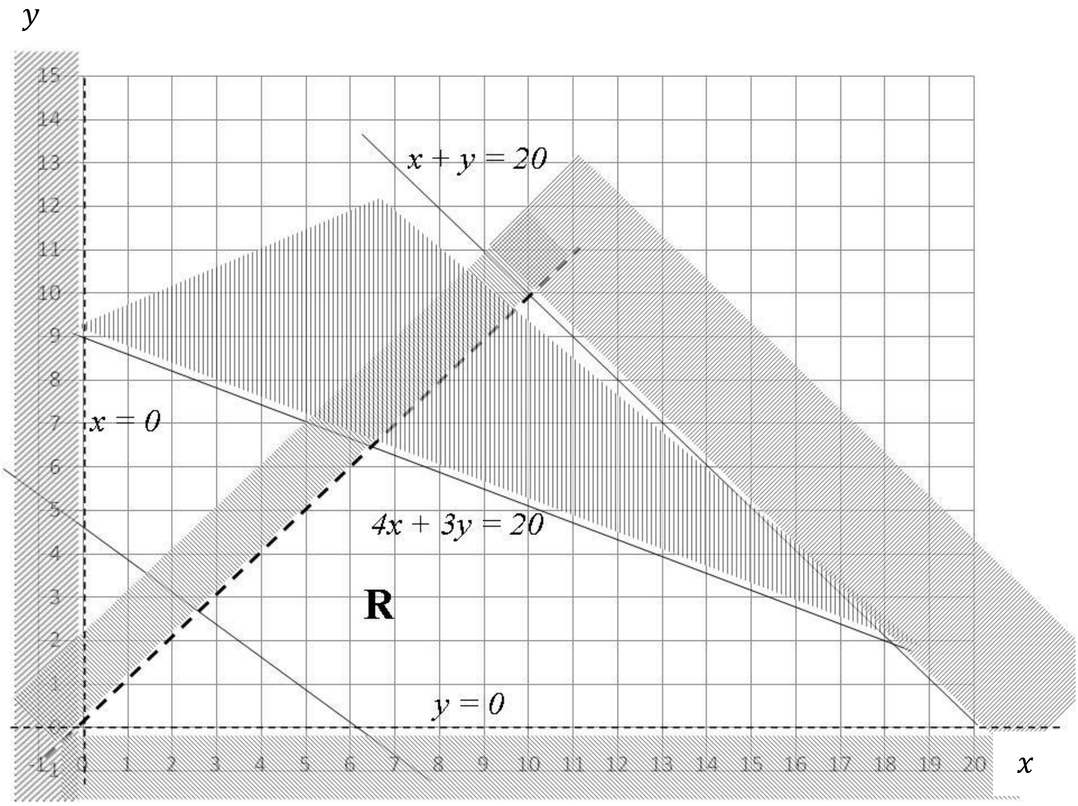
$x > y\,;\, x = y$

iv) For

$x > 0;\, x = 0$

v) For

$y > 0;\, y = 0$



The unshaded region represents the inequalities.

We now need to determine the objective to be achieved.

In this question, we need to find the maximum profit the farmer can make from the crops in a year.

Profit is given by the function $2\,000x + 3\,000y$, we call this the **objective function**. $\mathbf{2000x}$ represents profit from maize while $\mathbf{3000y}$ represents profit from beans.

We therefore need to get a suitable combination of x and y from the graph that will give us the maximum profit. We can do this by trying out the various co-ordinates that fall within the region R to see which of the co-ordinates will maximize the profit. In doing this, we can include all the points lying alongside the $lines\ 4x + 10y = 20, x + y = 20$ but do not include those points that lie along the lines $y = x, y = 0, and\ x = 0$ (broken lines). In many cases, trying out each of the points can take a lot of time. We therefore use a **search line** to help us identify the point.

In drawing a search line, we have to consider the objective function. i.e. $2000x + 3000y$.

The equation of the search line would be $2000x + 3000y = k$. To get the value $\boldsymbol{k}$ let us pick or select any point at random within the region R. e.g the point (5,1). Substitute this co-ordinate in the equation $2000x + 3000y = k$, we get the equation

$2000(5) + 3000(1) = k,$

$k = 13\ 000.$

The equation of the search line would therefore be $2000x + 3000y = 13\ 000$.

$2x + 3y = 13$(Simplified)

Draw the search line $2x + 3y = 13$

x	0	2	5	8
y	4.3	3	1	-1

To get the maximum point we move the search line parallel to itself further away from the origin (0, 0) until the last point.

We use a rule and a set of square to move the search line. In this case, the furthest point is the point (19, 1). This is the point that will give us maximum profit.

Maximum profit = (2 000×19) + (3 000×1)
= SSP 41 000

Note:

The point selected must have integer co-ordinates, i.e. the value of x and y must be integers.

To obtain minimum value of an objective function we move the search line parallel to itself towards the origin. The point closer to the origin is the minimum point.

Exercise 2

1. Draw the region represented by the following inequalities.

   $$2x + y \geq 8, y \leq 4, x - 2y \leq 0.$$

   Write down all the possible solution containing whole numbers only.

2. Mr. Deng makes 2 types of shoes A and B. He takes 2 hours to complete one pair of type A and 5 for a pair of type B. He takes a maximum of 120 hours to make x pairs of type A and y pairs of type B.
   It costs him SSP 400 to make a pair of type A and SSP 200 to make a pair of type B. His total cost does not exceed SSP 10 000. He must make 8 pairs of type A and more than 12 pairs of type B.
   a) Write down 4 inequalities representing the information.
   b) Draw a graph to represent the inequalities.
   c) If Mr. Deng makes a profit of SSP 50 on each pair of type A and SSP 70 on each pair of type B shoes, use the graph to determine the maximum possible profit he makes.

3. A shop keeper has sufficient money to buy a total of 100 crates of soft drink of types x and y. He wants to buy at least twice as many crates of type x as type y. He wants to buy maximum 80 crates of type x and at least 20 crates of type y.
   a) Write down all the inequalities to represent this statement.
   b) Show these inequalities on a graph and outline the region in which (x, y) must lie.
   c) The profit on a crate of type x is SSP 50 and that on a crate of type y is SSP 40.Find the number of crates of each type that he should buy to get maximum profit.
   d) Calculate his maximum profit.

4. A secondary school plans to take 529 students for a tour. There are 2 types of buses available, type x and type y. Type x can carry 68 passengers and type y can carry 44 passengers. They have to use at least 6 buses.
   The charges for hiring the buses are SSP 30 000 for type x and SSP 20 000 for type y.
   a) Find all the linear inequalities which will represent the above information.

b) On the grid provided, draw the inequalities and shade the unwanted region.

c) Use your graph above to determine the number of buses of each type that should be hired to minimize the cost.

Permutation and Combination formula

- **Permutation Formula**

  Permutation is defined as arrangement of *r* things that can be done out of total n things. This is denoted by $^{n}P_{r}$ which is equal to n!/(n-r)!

- **Combination formula**

  1. Combination is defined as selection of r things that can be done out of total n things. This is denoted by $^{n}C_{r}$ which is equal to n!/r!(n-r)!
  2. As per the Fundamental Principle of Counting, if a particular thing can be done in *m* ways and another thing can be done in *n* ways, then either one of the two can be done in $m + n$ ways and both of them can be done in $m \times n$ ways.

Example 1

How many four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6 (Repetition of digits not allowed)?

Solution:

Thousand's place can be filled in 6 ways. Hundred's place can be filled in 5 ways. Ten's place can be filled in 4 ways. Unit's place can be filled in 3 ways. So, using the Fundamental Principle of Counting, we get the answer as $6 \times 5 \times 4 \times 3 = 360$. Or using formula of Permutations, we need to arrange 4 digits out of total 6 digits. This can be done in $^{6}P_{4} = 360$ **ways.**

Example 2

There are 10 questions in an exam. In how many ways can a person attempt at least one question?

Solution:

A person can attempt 1 question or 2 questions or till all 10 questions. One question out of ten questions can be attempted in $^{10}C_{1} = 10$ ways. Similarly two questions out of ten questions can be attempted in ${}^{10}C_2 = 45$ ways. Going ahead by the same logic, all ten questions can be attempted in ${}^{10}C_{10} = 1$ way. Hence the total number of ways = 10 + 45 + 120 +.....10 + 1 = 1023 ways (Using the formula of Combination).

Exercise 3: Work in groups.

1. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

A. 24 400 B. 21 300

C. 210 D. 25 200

2. In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

A. 159 B. 209

C. 201 D. 212

3. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there in the committee. In how many ways can it be done?

A. 624 B. 702

C. 756 D. 812

4. In how many different ways can the letters of the word 'OPTICAL' be arranged so that the vowels always come together?

A. 610 B. 720

C. 825 D. 920

5. In how many different ways can the letters of the word 'CORPORATION' be arranged so that the vowels always come together?

A. 47 200 B. 48 000

C. 42 000 D. 50 400

6. In how many ways can a group of 5 men and 2 women be made out of a total of 7 men and 3 women?

A. 1 B. 126

C. 63 D. 64

7. In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels must always come together?

A. 9800 B. 100 020

C. 120 960 D. 140 020

8. There are 8 men and 10 women and you need to form a committee of 5 men and 6 women. In how many ways can the committee be formed?

A. 10 420 B. 11

C. 11 760 D. None of these

9. How many 3-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

A. 720 B. 420

C. None of these D. 5 040

10. In how many different ways can the letters of the word 'LEADING' be arranged such that the vowels should always come together?

A. None of these B. 720

C. 420 D. 122

Vectors

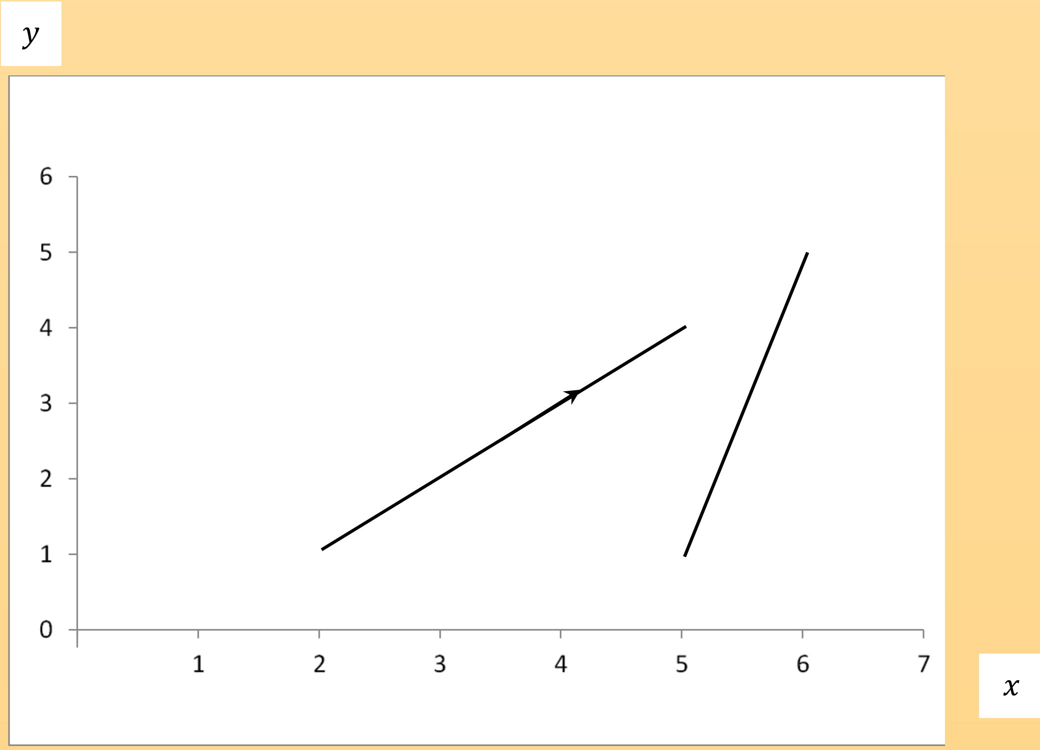
A vector is a quantity that has both **direction** and **magnitude**.

Consider the 2 statements below:

i) Juba is North West of Nairobi.
ii) Juba is 1100 km North West of Nairobi.

The first statement does not qualify to be called a vector because it lacks magnitude.

The second statement has both magnitude and direction. It is a vector quantity.



Vector $\overrightarrow{AB}$ can be written as a column vector ie. $\overrightarrow{AB}=\binom{3}{3}$.

To obtain vector AB we consider the coordinates of points A and B ie. A (2, 1) and B (5,4).

.$\overrightarrow{AB}=\binom{5}{4}$ - $\binom{2}{1}$

$= \binom{3}{3}$

Given points P $(x1 , y1)$ and Q $(x2 , y2)$, the vector $\overrightarrow{PQ} = \binom{x_2}{y_2} - \binom{x_1}{y_1}$

Example

Given that point A has the coordinates (2, -1) and point B (3, 5).Find vector AB as a column vector.

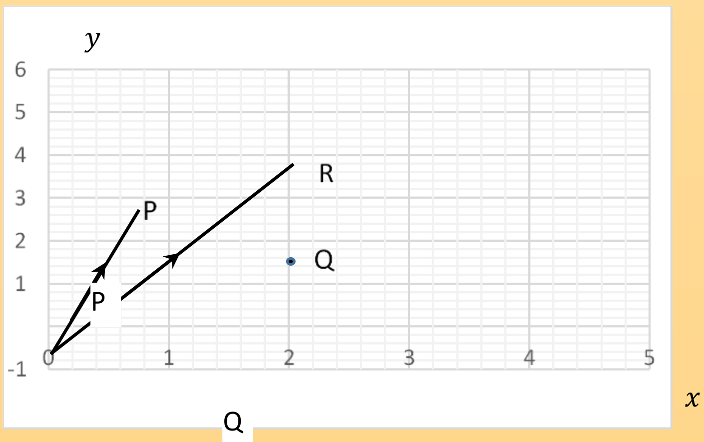
Solution

$\overrightarrow{AB} = \binom{3}{5} - \binom{2}{-1}$

$= \binom{1}{6}$

Position Vector

Position vector shows the displacement of a point from the origin (0, 0).



The position of P above is given by

$\overrightarrow{OP} = \binom{1}{3}$

Similarly position vector of R is given by

$\overrightarrow{OR} = \binom{3}{4}$

Mid-point

Given that A (x_1 , y_1) and B (x_2, y_2), the mid-point M of line AB is given by

M = $(\frac{x1+x2}{2}, \frac{y1+y2}{2})$

Example

Find the mid-point of line AB given that the position vectors of A and B are $\binom{2}{1}$ and $\binom{3}{6}$ respectively.

Solution

$\overrightarrow{OA} = \binom{2}{1}$ hence A = (2, 1)

$\overrightarrow{OB} = \binom{3}{6}$ hence B = (3, 6)

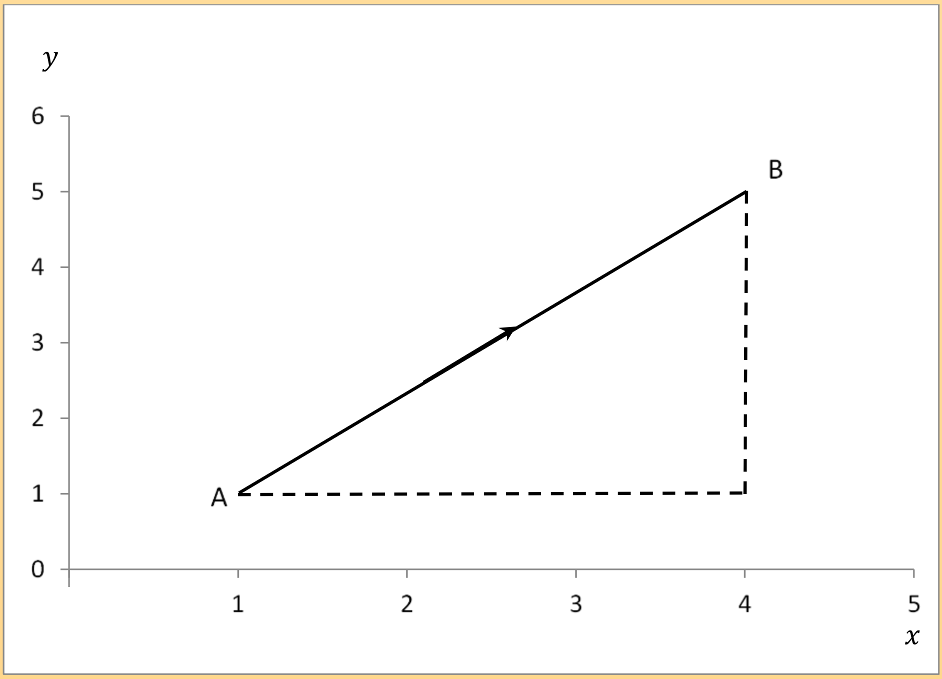
M = $(\frac{x1+x2}{2}, \frac{y1+y2}{2})$

M = $(\frac{2+3}{2}, \frac{6+1}{2})$

M = (2.5, 3.5)

Magnitude of a vector

The magnitude or modulus of a vector is the same as the length of a line.



The length of line AB can be obtained by;

$$|AB| = \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5 \text{ Units.}$$

We can therefore say that if $A\ (x_1, y_1)$ and $B\ (x_2\ , y_2)$then the magnitude of AB is given by

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the example above A = (1, 1) while B = (4, 5). The magnitude of AB is given by

$|AB| = \sqrt{(4-1)^2 + (5-1)^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 6}$

$= \sqrt{25}$

= 5 Units.

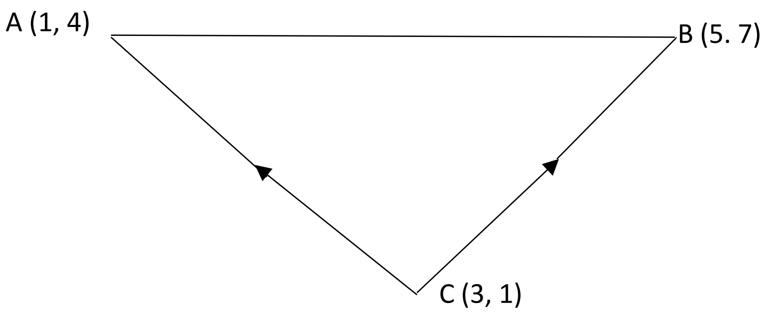
Exercise 4

1. Given the coordinates P (2, 1), Q (3, 0) and R (4, 7). Find the following vectors;
   a) $\overrightarrow{PQ}$
   b) $\overrightarrow{PR}$
   c) $\overrightarrow{RQ}$

2. Using the coordinates given in question no 1 above, find;
   a) |PQ|
   b) The mid-point of PQ

3. Given that $\overrightarrow{OA} = \binom{2}{6}$ and $\overrightarrow{OB} = \binom{4}{1}$; Find
   a) The position vector of the mid-point of line AB.
   b) The vector $\overrightarrow{BA}$
   c) The magnitude of AB.

4.



From the figure above, find;

a) The coordinates of the mid-point of AB.

b) The magnitude of MC.

Multiplication by a Scalar

If $\boldsymbol{a} = \begin{pmatrix}3\\2\end{pmatrix}$then $2\boldsymbol{a}$=$2\begin{pmatrix}3\\2\end{pmatrix}$= $\begin{pmatrix}6\\4\end{pmatrix}$

Each element is multiplied by 2.

Parallel Vectors

Vectors are said to be parallel if one is a multiple of the other e.g.

If $\overrightarrow{AB} = \begin{pmatrix}3\\2\end{pmatrix}$and $\overrightarrow{QR} = \begin{pmatrix}-9\\-6\end{pmatrix}$ then $\overrightarrow{AB}$ is parallel to $\overrightarrow{QR}$ since.

-3 $\overrightarrow{AB} = \overrightarrow{QR}$ or

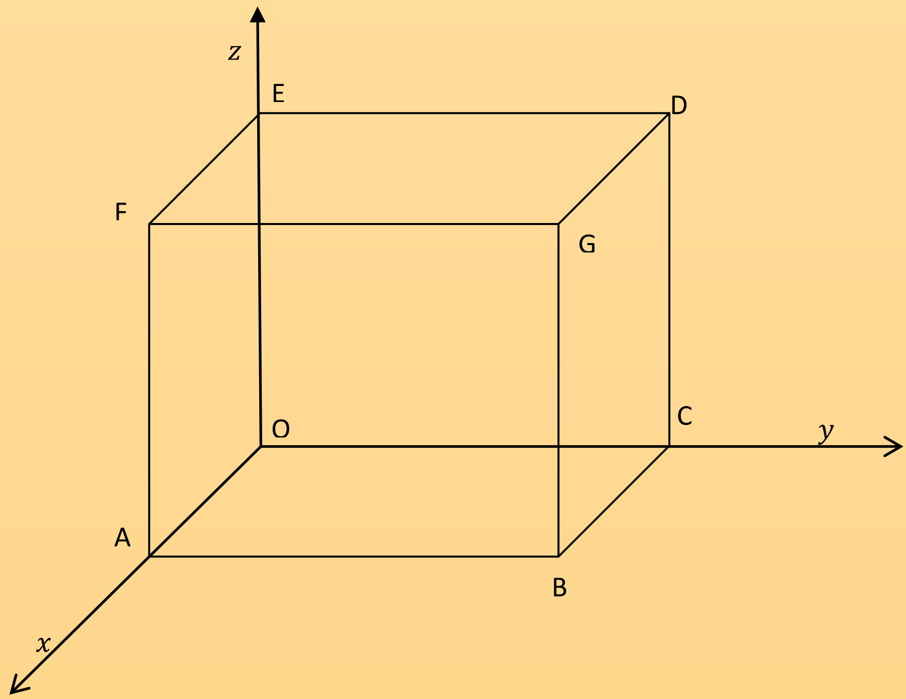
$\overrightarrow{AB} = \frac{-1}{3}\overrightarrow{QR}$

Exercise 5

1. Given that $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Find the value of the following in column vector form;
   a) 3**a** – **b**.
   b) (**c** – **b**)
   c) **a** + **b** + **c**.
   d) **c** + 2**d** +$\frac{1}{2}$**a**.
   e) 3**c** – 2**d** + **a**.
   f) 4**a** – 3**c** + $\frac{1}{2}$**d**.
2. Use the values of a, b, c and d in question 1 above to find the values of x in each of the following cases.
   a) $3x + \boldsymbol{b} = \boldsymbol{c}$
   b) $x - \boldsymbol{b} = \boldsymbol{a}$
   c) $2\boldsymbol{x} + \boldsymbol{c} = 0$
   d) $\boldsymbol{b} = 2x - \boldsymbol{a}$
   e) $x = \boldsymbol{b} + \boldsymbol{c}$
   f) $3\boldsymbol{a} + \boldsymbol{b} = x - \boldsymbol{d}$
3. State whether the following are 'true' or 'false'
   a) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
   b) $\begin{pmatrix} 5 \\ 15 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -3 \end{pmatrix}$
   c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is parallel to $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
   d) $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ is parallel to $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$
   e) $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is parallel to $\begin{pmatrix} -12 \\ 4 \end{pmatrix}$

f) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ is parallel to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Co-ordinates in Three Dimensions



The figure above (not drawn to scale) shows the model of a cuboid.

Taking OA as the $x - axis$, OC as the $y - axis$ and OE as the Z axis, we can give the co-ordinates of various points on this diagram in 3 dimension. i.e. In terms of (x, y, z)

Assuming point H lies 3 units along the x-axis, 2 units along the y-axis and 0 units along the z-axis then the coordinates of H would be

H = (3, 2, 0)

Similarly;

Point C = (0, 6, 0)

Point F = (3,0, 5)

Point G = (3, 6, 5)

Column and position vectors in 3 dimensions.

Given the coordinates of A = (5, 1, 1) and B = (6, 5, 3). The position vectors of A and B are

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$$

Example

Given the position vectors of Q and R are $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ respectively, find vector $\overrightarrow{QR}$

$$\overrightarrow{QR} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$$

Magnitude of a vector in 3 Dimensions

If A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2) then the magnitude of **AB** is given by

$$|\text{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_{2)}^2 + (z_1 - z_2)^2}$$

Example

Find the magnitude of vector AB given that A = (3, 6, 2) and B = (-2, 1, 6)

Solution

$$|AB| = \sqrt{(-2-3)^2 + (1-6)^2 + (6-2)^2}$$

$$= \sqrt{(-5)^2 + (-5)^2 + (4)^2}$$

$$= \sqrt{66}$$

$$= 8.124\ units$$

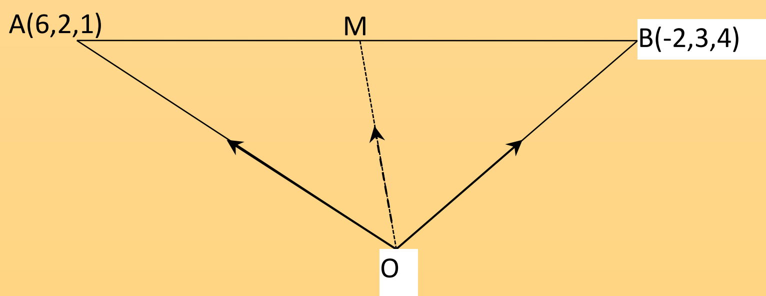
Mid point of a vector in 3 Dimensions

The midpoint of A = (x_1, y_1, z_1) and B =(x_2, y_2, z_2)is given by

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example given that the position vector of A and B are $\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ respectively, find the position vector of midpoint of $\overrightarrow{AB}$

Solution



$M=(\frac{6+-2}{2} + \frac{2+3}{2} + \frac{1+4}{2})$

M= (2, 2.5, 2.5)

$$\overrightarrow{OM} = \begin{pmatrix} 2 \\ 2.5 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.5 \\ 2.5 \end{pmatrix}$$

Exercise 6

1. Find the modulus of each of the following vectors.

   a) $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$

   b) $\begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}$

   c) $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

2. Given that $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$ and and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$Find

   i) |AB|

   ii) The position vector of **c**, the mid-point of line AB.

3. Given that $a = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}$ $and\ c = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$find the value of each of the following;

   (i) 2**a** + **b** – **c**

   (ii) 2**b** – **c**

   (iii) ½ **a** – **b**+ **c**

   (iv) **a** + 3**b** – 2**c**

   (v) 2**c** – 3**a** +**b**

4. Given that $\boldsymbol{a} = \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ find the value of x

where,

(i) $\boldsymbol{6x} + \boldsymbol{b} = \boldsymbol{c}$

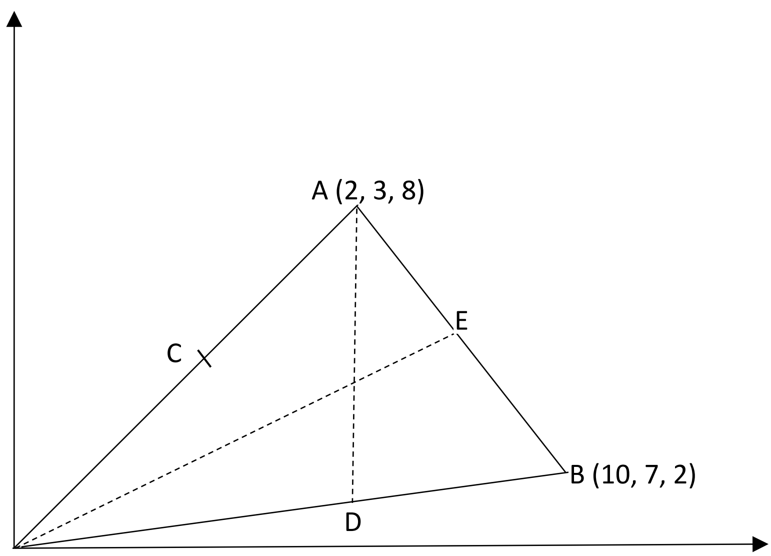
(ii) $2\boldsymbol{x} + \boldsymbol{b} = 3\boldsymbol{c}$

(iii) $\boldsymbol{a} + \boldsymbol{b} = 5x$

(iv) $3x - \boldsymbol{c} = \boldsymbol{b}$

(v) $\boldsymbol{a} + x = 3\boldsymbol{c}$

(vi) $\frac{1}{2}\boldsymbol{a} - x = \boldsymbol{b} + \boldsymbol{c}$



5. The figure above shows triangle OAB. A is the point (2, 3, 8) and B (10, 7, 2) C, D and E are the mid- point of OA, OB and AB respectively

a) Find;

(i) The co-ordinates of C and D

(ii) The length of the vector CD and AB

b) Show that

(i) CD is parallel to AB

(ii) DE is parallel to OA

6. The position vectors of points A and B are and respectively.

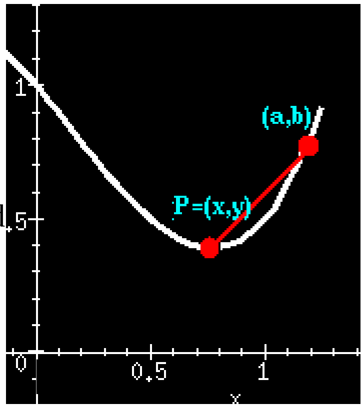
A point Q divides AB in the ratio 2:1. Find the position vector of Q

UNIT 4

CALCULUS

The essence of calculus is the **derivative**. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the **tangent line** to the function at a point. Let's use the view of derivatives as tangents to motivate a geometric definition of the derivative.

We want to find the slope of the tangent line to a graph at the point *P*. We can approximate the slope by drawing a line through the point *P* and another point nearby, and then finding the slope of that line, called a **secant line**. The slope of a line is determined using the following formula (*m* represents slope) :



$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}.$$

Let *P* = (*x*,*y*) and *Q* := (*a*,*b*). Let

$$a = x + \Delta x \quad \text{and} \quad b = y + \Delta y = f(a) = f\left(x + \Delta x\right).$$

Then the slope of the line

$$\overline{PQ} \quad \text{is} \quad \frac{b-y}{a-x} = \frac{\Delta y}{\Delta x} = \frac{f\left(x+\Delta x\right) - f(x)}{\Delta x}.$$

Now, we chose an arbitrary interval to be Delta-*x*. How does the size of Delta-*x* affect our estimate of the slope of the tangent line? The smaller Delta-*x* is, the more accurate this approximation is.

What we want to do is decrease the size of Delta-*x* as much as possible. We do this by taking the limit as Delta-*x* approaches zero. In the limit, assuming the limit exists, we will find the exact slope of the tangent line to the curve at the given point. This value is the derivative;

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

There are a few different, but equivalent, versions of this definition. In more general considerations, *h* is often used in place of Delta-*x*. Or Delta-*y* is used in place of

$$f(x + \Delta x) - f(x).$$

This leads to three commonly used ways of expressing the definition of the derivative:

$$\begin{aligned} \frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \end{aligned}$$

Derivative of a polynomial

Polynomials are some of the simplest functions we use. We need to know the derivatives of polynomials such as x^4+3x, $8x^2+3x+6$, and 2. Let's start with the easiest of these, the function $y=f(x)=c$, where c is any constant, such as 2, 15.4, or one million and four (10^6+4). It turns out that the derivative of any constant function is zero. This makes sense if you think about the derivative as the slope of a tangent line. To use the definition of a derivative, with $f(x)=c$,

$$\begin{aligned}\frac{d}{dx}(c) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= 0.\end{aligned}$$

For completeness, now consider $y=f(x)=x$. This is the equation of a straight line with slope 1, and we expect to find this from the definition of the derivative. We are not disappointed:

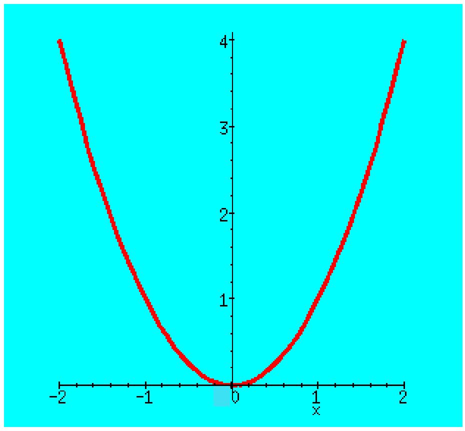
$$\begin{aligned}\frac{dy}{dx} = \frac{dx}{dx} &= \lim_{h\to 0}\frac{f(x+h)-x}{h}\\ &= \frac{(x+h)-x}{h}\\ &= \frac{h}{h}\\ &= 1.\end{aligned}$$

Two things to note in the above:

- It may be tempting to ``cancel" the term ``*dx*" in the intermediate step. This is valid, but only in this simple case.
- It will never be as easy as this again, although it won't be much harder.

  Before going to the most general case, consider $y=f(x)=x^2$. This is the most basic parabola, as shown. The derivative of $f(x)$ may still be found from basic algebra:

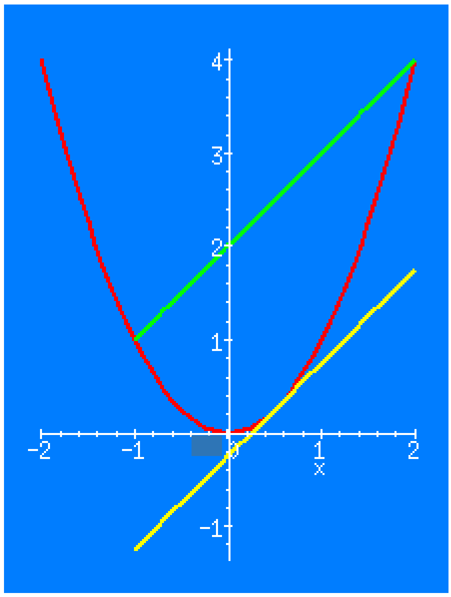
$$
\begin{aligned}
\frac{dy}{dx} = \frac{dx}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2\,x\,h + h^2) - x^2}{h} \\
&= \lim_{h\to 0} \frac{2\,x\,h + h^2}{h} \\
&= \lim_{h\to 0} (2\,x + h) \\
&= 2\,x.
\end{aligned}
$$



This tells us exactly what we expect; the derivative is zero at *x*=0, has the same sign as *x*, and becomes steeper (more negative or positive) as *x* becomes more negative or positive.

An interesting result of finding this derivative is that the slope of the secant line is the slope of the function at the midpoint of the interval. Specifically,

$$
\frac{\Delta y}{\Delta x} = 2x + h = 2\left(x + \frac{h}{2}\right) = f'\left(x + \frac{h}{2}\right).
$$



(In the figure shown, *x* = -1 and *h* = 3, so (*x*+*h*/2) = +1/2.
Please note that parabolic functions are the *only* functions (aside from linear or constant functions) for which this is always true.

From here, we can and should consider $y=f(x)=x^n$ for any positive integer *n*. There are many ways to do this, with varying degrees of formality.

To start, consider that for *n* a positive integer, the binomial theorem allows us to express *f*(*x*+h) as

$$\begin{aligned} f(x+h) &= (x+h)^n \\ &= x^n + n\,x^{n-1}\,h + (n(n-1)/2)\,x^{n-2}\,h^2 + \cdots \\ &\quad + n\,x\,h^{n-1} + h^n. \end{aligned}$$

(In the above, there will always be no more than n+1 nonzero terms.) Then, algebra again gives us

$$\begin{aligned} \frac{d}{dx}\,x^n &= \lim_{h\to 0} \frac{\left(x^n + n\,x^{n-1}\,h + \cdots + + h^n\right) - x^n}{h} \\ &= \lim_{h\to 0} \left(n\,x^{n-1} \cdots + h^{n-1}\right) \\ &= n\,x^{n-1}. \end{aligned}$$

This very convenient form is seen to reproduce the above results for n=1, n=2 and even n=0, which is the case c=1.
The above result could be found from an inductive process, using the product rule, but the inductive step is similar to that which allows extension of the binomial theorem to all positive integers, and adds little to this presentation.

The extension from $f(x) = x^n$ to arbitrary polynomials (only finite order will be considered here) needs only two straightforward, perhaps even obvious results:

- The derivative of the sum of two function is the sum of the derivatives.
- The derivative of a function multiplied by a constant is the derivative of the function multiplied by the same constant.

In symbols, these results are

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d\,f}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}\left(c\,f(x)\right) = c\frac{d\,f}{dx}.$$

In the above, c is a constant, and differentiability of the functions at the desired points is assumed.

Combining all of these results, we can see that for the coefficients a_k all constants,

$$\begin{aligned} \text{If} \qquad & f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3x^3 + \cdots \\ & \qquad + a_{m-1}\,x^{m-1} + a_m x^m \\ \text{then} \qquad & \frac{d}{dx}f(x) = f'(x) = a_1 + 2a_2\,x + 3a_3\,x^2 + \ldots \\ & \qquad + (m-1)a_{m-1}x^{m-2} + ma_m x^{m-1}. \end{aligned}$$

This is often seen in summation notation as

$$\begin{aligned} \text{If} \qquad & f(x) = \sum_{k=0}^{m} a_k\,x^k \\ \text{then} \qquad & \frac{d}{dx}f(x) = f'(x) = \sum_{k=0}^{m} ka_k\,x^{k-1}. \end{aligned}$$

Example

1. If $y = 3x^2$
   $y' = 6x$

2. If $y = x^4$
   $y' = 4x^3$

3. If $y = 3x^4 + x^3$
   $y' = 12x^3 + 3x^2$

4. If $y = -4x^2 + 6x - 3$
   $y' = -8x + 6$

Exercise 1

1. Find the derivative of the following.
   a) $y = 6x^2$
   b) $y = \frac{1}{3}x^6$
   c) $y = \frac{2}{3}x^3$
   d) $y = -\frac{1}{6}x^{12}$
   e) $y = 8$
   f) $y = 6x + 1$
   g) $s = 2t^3 + 4t^2 - 6t + 5.$
   h) $s = -6t^5 + 5t^4 - t^3.$
2. Find the derived function in each of the following cases.
   a) $y = 3x^6 + \frac{1}{4}x^3 - x.$
   b) $y = 20x^3 - \frac{1}{4}x^4$
   c) $y = \frac{1}{4}x^3 - 3x^2 + \frac{5}{2}x.$
   d) $y = x(5x + 3) + 4(x + 1)^2.$

Equation of a tangent and normal to the curve at a point.

Tangent and Normal Lines

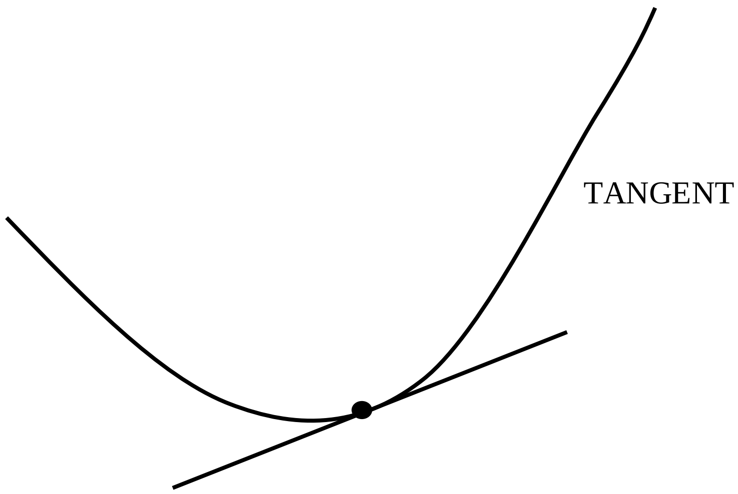
Example 1

For a given curve of equation $y = 3x^2 + 2x + 1$

$y' = 6x + 2$. This is the gradient function.

To find the gradient of this curve at a point, say $(2, 3), = 6 \times (2) + 2 = 14.$

The gradient of a curve at a point is the same as the gradient of a tangent to the curve at the same point.



Example 2

Find the gradient of the curve $y = x^3 + 2x^2 + 3x$ at the point $(1, 4)$.

Solution

$$y = x^3 + 2x^2 + 3x.$$

$$y' = 3x^2 + 4x + 3.$$

Gradient at$(1, 4) = 3 + 4 + 3 = 10.$

This is the gradient of the tangent to this curve at the point $(1,4)$.

To get the equation of the tangent, we create an arbitrary point through which it passes, (x, y).

Gradient $= \frac{\Delta y}{\Delta x}$;

$$\frac{10}{1} = \frac{y-4}{x-1}$$

$y - 4 = 10x - 10$

$y = 10x - 6.$

A normal to curve is a straight line perpendicular to the tangent.

To get the equation of the normal,

$$\frac{y-4}{x-1} = -\frac{1}{10}$$

$10y - 40 = 1 - x$

$10y = 41 - x.$

$$y = \frac{41 - x}{10}.$$

Example 3

Find the equation of the tangent and normal to the curve.

$y = x^3 + 5x^2 - 3x + 2$ at $(1, 0)$

$y' = 3x^2 + 10x - 3$

at $(1, 0)$gradient $= 3 + 10 - 3$

$= 10$

Equation of the tangent at (1, 0)

$$\frac{y-0}{x-1} = \frac{10}{1}$$

$$\frac{y}{x-1} = \frac{10}{1}$$

$y = 10x - 10.$

Equation of normal at (1, 0)

$$\frac{y-0}{x-1} = -\frac{1}{10}$$

$$\frac{y}{x-1} = -\frac{1}{10}$$

$$10y = 1-x$$

$$y = \frac{1-x}{10}$$

Exercise 2

1. Find the equation of the tangent to the curve $y = 3x^2$ at $(2, 8)$.
2. Find the equation and normal to the curve $y = (x^2 + 1)(x-2)$ when $x = 2$.
3. Given the equation of a curve is $y = 5x^3 - 7x^2 + 3x + 1$. Find the gradient and equation of the tangent at (1,3).
4. The equation of a curve is given by $y = x^3 - 4x^2 - 3x$.
   a) Find the value of y, when $x = -1$.
   b) Find the equation of the normal to the curve at $x = 1$.

Stationary points: Maximum and Minimum Values

Differentiation can be used to find the maximum and minimum values of a function. Because the derivative provides information about the gradient or slope of the graph of a function we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximising power, or profit, or minimising losses or costs.

When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of variables. Then, functions are used to describe the ways in which these variables change. A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values, its turning points. Drawing a graph of a function using a graphical calculator or computer graph plotting package will reveal this behaviour, but if we want to know the precise location of such points we need to turn to algebra and differential calculus. In this section we look at how we can find maximum and minimum points in this way.

Consider the graph of the function, $y(x)$, shown in Figure 1. If, at the points marked A, B and C, we draw tangents to the graph, note that these are parallel to the x axis. They are horizontal. This means that at each of the points A, B and C the gradient of the graph is zero.

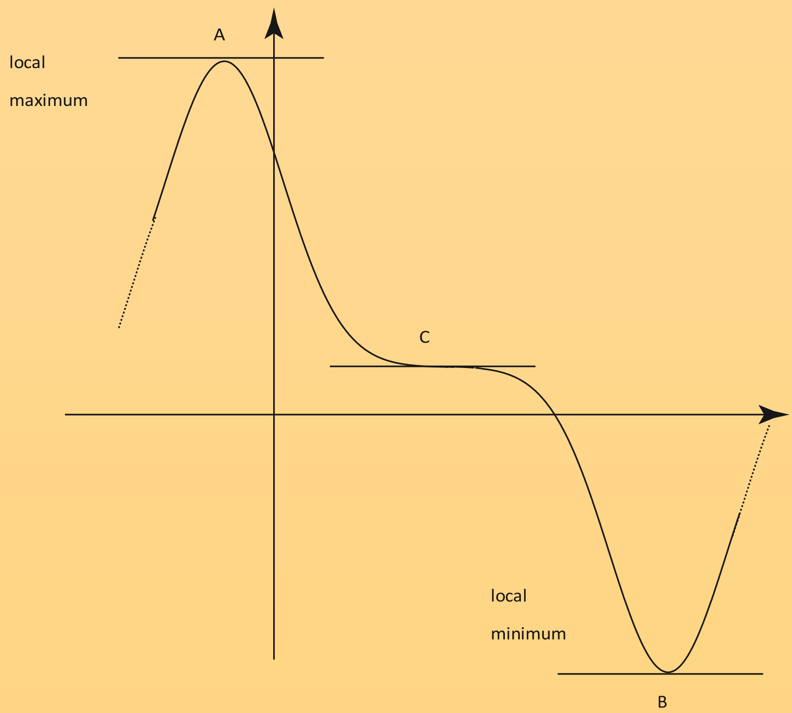


Figure 1. The gradient of this graph is zero at each of the points A, B and C.

We know that the gradient of a graph is given by $\frac{dy}{dx}$ consequently, $\frac{dy}{dx} = 0$ at points A, B and

C. All of these points are known as stationary points.

Key point

Any point at which the tangent to the graph is horizontal is called a stationary point. We can locate stationary points by looking for points at which $\frac{dy}{dx} = 0$.

Turning points

Refer again to Figure 1. Notice that at points A and B the curve actually turns. These two stationary points are referred to as turning points. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points.

But not all stationary points are turning points (e.g. point C).

In other words, there are points for which $\frac{dy}{dx} = 0$ which are not turning points.

Not all points where __ = 0 are turning points, i.e. not all stationary points are turning points.

Point A in Figure 1 is called a local maximum because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function. Point B in Figure 1 is called a local minimum because in its immediate area it is the lowest point, and so represents the least, or minimum, value of the function. Loosely speaking, we refer to a local maximum as simply a maximum. Similarly, a local minimum is often just called a minimum.

Distinguishing maximum points from minimum points

Think about what happens to the gradient of the graph as we travel through the minimum turning point, from left to right, that is as x increases. Study Figure 2 to help you do this.

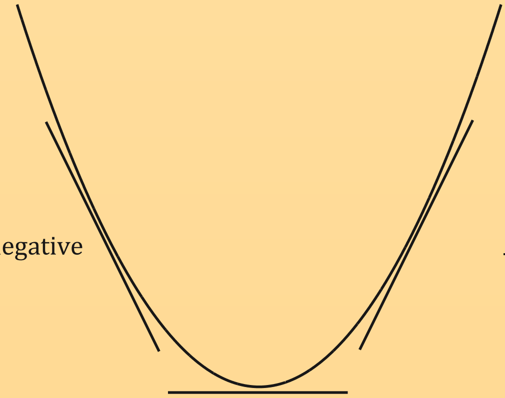


Figure 2. $\frac{dy}{dx}$ goes from negative through zero to positive as x increases.

Notice that to the left of the minimum point, $\frac{dy}{dx}$ is negative because the tangent has negative gradient. At the minimum point, $\frac{dy}{dx} = 0$. To the right of the minimum point $\frac{dy}{dx}$ is positive, because here the tangent has a positive gradient. So, $\frac{dy}{dx}$ goes from negative, to zero, to positive as x increases. In other words, ___ must be increasing as x increases.

In fact, we can use this observation, once we have found a stationary point, to check if the point is a minimum. If $\frac{dy}{dx}$ is increasing near the stationary point then that point must be minimum.

Now, if the derivative of $\frac{dy}{dx}$ is positive then we will know that $\frac{dy}{dx}$ is increasing; so we will know
that the stationary point is a minimum. Now the derivative of $\frac{dy}{dx}$, called the second derivative,
is written $\frac{d^2y}{dx^2}$ We conclude that if $\frac{d^2y}{dx^2}$ is positive at a stationary point, then that point must be a minimum turning point.

Example 1

The table below shows values of the gradient at different points for the curve.

$y = 2x^2 + 5$

$y' = 4x$

x	-5	-4	-3	-2	0	1	2	4	5	
y'	-20	-16	-12	-8	0	4	8	16	20	

Observation

- Gradient at $x = 0$ is zero.
- Gradient to the left of $x = 0$ is negative.
- Gradient to the right of $x = 0$ is positive.

Example 2

The table below shows values of gradient at different points for the function

$y = -x^2 + 4x + 6.$

x	-3	-2	0	1	2	3	4	5	6
y'	10	8	4	2	0	-2	-4	-6	-8

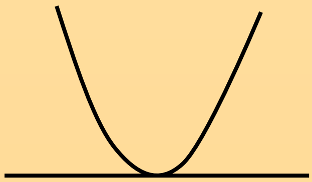
Observation

- Gradient at $x = 2$ is positive.
- Gradient to the left of $x = 2$ is positive.
- Gradient to the right of $x = 2$ is negative.

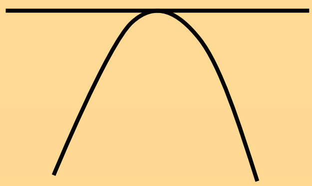
NOTE

A point on the curve at which the gradient is zero is a stationary (turning) point. Such a point can be a minimum or maximum value.

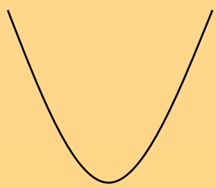
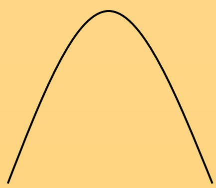
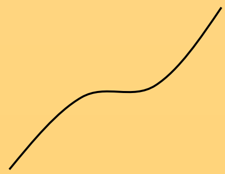
A minimum point is one at which the gradient changes from negative through to positive as in the first example above.

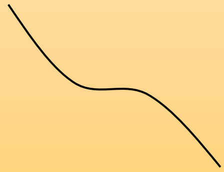


A maximum point is one at which the gradient changes from positive through to negative.



A turning point at which the gradient remains positive or negative even after the zero point is called a point of inflexion.

Type of Stationary point.	**Gradient to the left**	**Gradient to the right**	**Diagram Representation.**
Minimum	Negative	Positive	
Maximum	Positive	Negative	
Inflexion	Positive	Positive	

Inflexion	Negative	Negative	

Example

1. Identify the type of stationary points in the following curves.

   a) $y = 2x^3 - 6x + 2.$

   $y' = 6x^2 - 6$

   $6x^2 - 6 = 0$

   $6x^2 = 6$

   $x^2 = \frac{6}{6}$

   $x^2 = 1$

   $x = +1$ or -1

   For $x = 1.$

x	-2	1	3
y'	18	0	48
$Nature\ of\ Gradient$	$+$	0	$-$

Point of inflexion.

For $x = -1$

x	-2	-1	0
y'	18	0	-6
$Nature\ of\ Gradient$	$+$	0	$-$

Maximum point.

Exercise 3

1. Find the nature of the turning points in the following curves.
   a) $y = 3x^2 + 12x$
   b) $y = 2x^2 + \frac{1}{3}x^3$
   c) $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x$
   d) $y = 5x - 6 - x^2$
   e) $y = 6 - 5x + x^2$
   f) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$

Application to Kinematics: Calculation of velocity and acceleration.

Velocity

Gradient $= \frac{\Delta y}{\Delta x}$

For a displacement-time graph therefore,

$$\text{gradient} = \frac{\Delta\, displacement}{\Delta\, time}$$

This imply that the gradient function of a displacement-time graph is given by

$$\frac{ds}{dt} = Velocity$$

Activity 1: Work in groups

Purpose: To relate a graphical plot of a student's change in position with the actual change in position along a number line.

Procedure

1. Place 11 small pieces of electrical tape (about 3 inches long) at 1-m intervals in a straight line along the floor.

2. Make 11 -3” x 5” index cards labelled, 5 m, 4 m, 3 m, 2 m, 1 m, 0 m, -1 m, -2 m, -3 m, -4 m, -5 m, and place them in order at the 11 tape location.
3. Have a student start at 0 m, then move to +2 m, then +5 m, then to +3 m, then to -1 m, then to -3 m, then stay at -3 m, and finally go to 0 m.
4. Plot a graph of the student’s location (in metres) as a function of event (7 in this case). Evenly space the event numbers to represent actual times of each event.
5. Connect the data points using a straight line between the points 0 m to 1 m, 1 m to 5 m, and so on.
6. Study the complete graph of location vs event and discuss what is happening from start to finish.

Examples

1. The displacement of a car from a fixed point after a time t seconds is given by $S = 18t + 4$. Calculate the velocity at $t = 3$.

Solution

$S = 18t + 4.$

$\frac{dS}{dt} = 18$

This car is moving at a constant velocity of $18m/s$.

2. The displacement, S meters of a moving object after t seconds is given by $S = t^3 + 3t^2 - 8t + 2$. Find the velocity at

a) t = 2.

Solution

$\frac{dS}{dt} = 3t^2 + 6t - 8.$

$= 3 \times 4 + 6 \times 2 - 8$

$= 12 + 12 - 8$

$= 16m/s.$

b) $t = 5$.

Solution

$$\frac{dS}{dt} = 3t^2 + 6t - 8.$$

$$= 3 \times 25 + 6 \times 5 - 8.$$

$$= 75 + 30 - 8$$

$$= 97m/s.$$

3. An object moves such that its displacement S meters is S = $t^3 + 3t^2 + 5$. Find its velocity at:
   a) $t = 0.$

      $\frac{ds}{dt} = 3t^2 + 6t.$

      $= 0m/s.$

   b) $t = 3.$

      $$\frac{ds}{dt} = 3t^2 + 6t$$

      $$= 3 \times 9 + 6 \times 3$$

      $$= 27 + 18$$

      $$= 45m/s.$$

Activity 2: Work in groups

Displacement

1. Draw to scale and solve:
   a) Deng goes 8 steps north, 3 steps east, 6 steps south, 6 steps west and 2 steps south. What is John's displacement from his starting point?
   b) Keji hikes east 6 kilometres, north 3 kilometres and south 8 kilometres. What is Keji's displacement from her starting point?

Acceleration

In a velocity-time graph, the gradient is given by

$$\frac{\Delta y}{\Delta x} = \frac{\Delta\, Velocity}{\Delta\, Time} = \frac{dV}{dt}$$

Therefore, the derivative of a velocity-time equation is acceleration.

Example

The velocity of a moving particle after t seconds is given by v = t^2 + 10t – 3. Find the acceleration at

a) $t = 1\, Sec.$

$\frac{dV}{dt} = 2t + 10$

$= 2 \times 1 + 10$

$= 2 + 10$

$= 12m/s^2$

b) $t = 3.$

$\frac{dV}{dt} = 2t + 10.$

$= 2 \times 3 + 10.$

$= 6 + 10.$

$= 16m/s^2.$

At what instants are the velocity and acceleration zero in a motion given by

$V = 5t^2 - 10t.$

Solution.

i) Velocity.

$0 = 5t^2 - 10t.$

$0 = 5t\,(t - 2).$

$t = 0\, or\, 2.$

ii) Acceleration.

$\frac{dv}{dt} = 10t - 10.$

$10t - 10 = 0$
$10t = 10$
$t = 1.$

Exercise 4

1. An object moves upwards vertically such that at any time t seconds, its height h meters above the ground is given by $h = -5t^2 + 30t$.
   a) Calculate its velocity at $t = 1$, and $t = 3$.
   b) How far up does it travel?
2. A particle moves such that its distance from a fixed point is given by $S = t^3 - \frac{5}{2}t^2 + 2t - 5$ meters.
   a) Find its acceleration after t secs.
   b) Find its velocity when acceleration is zero.
3. The velocity of a moving object is given by $v = 5t^2 - 12t + 7$. Calculate its acceleration at $t = 2$.
4. The displacement S meters covered by a particle after t secs is given by $S = t^3 - 6t^2 + 9t - 4$.
   a) Calculate the velocity of the curve at $t = 1.5$ sec.
   b) Determine the value of S at the maximum turning point.
5. A stone is thrown upwards. After t seconds, its height from the ground is given by $h = 10 + 20t - 15t^2$. Find its height, velocity, and acceleration of the stone when;
   a) $t = 2$.
   b) $t = 3$.
   c) $t = \frac{5}{3}$.

Integration and Area under a curve

Integration is the reverse process of differentiation. It means obtaining the original equation from the gradient function.

For example, given a gradient function,$y' = 2$ there are several lines whose gradient is 2. Integration enables us to know which one of them was differentiated to get the 2.

Generally, if $y' or \frac{dy}{dx} = a^b \; then \; y = \frac{a^{b+1}}{b+1} + c$

Where c is a constant and $b \neq -1$

Examples

1. Integrate the following.

 a) $2x^3$

$$\frac{dy}{dx} = 2x^3$$

$$y = \frac{2x^4}{4} = \frac{1}{2}x^4 + c$$

 b) $3x^4$

$$\frac{dy}{dx} = 3x^4$$

$$y = \frac{3x^5}{5} + c$$

 c) $3x^2$

$$\frac{dy}{dx} = \frac{3x^3}{3} + c$$

$$= x^3 + c$$

 d) $5x^2 - 3x + 4.$

$$\frac{dy}{dx} = 5x^2 - 3x + 4$$

$$y = \frac{5x^3}{3} - \frac{3x^2}{2} + 4x + c$$

2. Find the general equation of a curve whose gradient function is:

a) $x^2 + 4x$.

$$\frac{dy}{dx} = x^2 + 4x$$

$$y = \frac{x^3}{3} + \frac{4x^2}{2} + c$$

$$= \frac{1}{3}x^3 + 2x^2 + c$$

b) $\frac{dy}{dx} = \frac{1}{x^2} + 5.$

$$= x^{-2} + 5.$$

$$y = \frac{x^{-1}}{-1} + 5x + c.$$

$$= -x^{-1} + 5x + c.$$

Area under a curve

Integration can also be used to find the area under a curve as follows:

Example 1

Calculate the area bounded by the curve $y = x^2 + 2x$, the line $x = 1$ and $x = 5$.

Solution

By integration, y = x^2 +2x.

$$A = \int_1^5 (x^2 + 2x)dx$$

$$= \left[\frac{1}{3}x^3 + x^2\right]_1^5$$

$$= \left(\frac{5^3}{3} + 5^2\right) - \left(\frac{1^3}{3} + 1^2\right)$$

$$= \left(\frac{125}{3} + 25\right) - \left(\frac{1}{3} + 1\right)$$

$$= \frac{200}{3} - \frac{4}{3}$$

$$= \frac{196}{3}$$

$= 65\ \frac{1}{3} sq\ units.$

Example 2

Calculate the area enclosed by the curve $y = x^2 + x$, the straight line $x = 1$ and $x = 3$.

$$A = \int_1^3 (x^3 - 3x^2 + 4x)dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^3$$

$$= \left(\frac{3^4}{4} - 3^3 + 3^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)$$

$$\left(\frac{81}{4} - 27 + 9\right) - \left(\frac{1}{4} - 1 + 1\right)$$

=2.25 sq. u.

Example 3

Calculate the area bounded by the curve $y = x^2 + x$, the straight lines $x = 2$ and $x = 4$

$$A = \int_2^4 x^2 + x. dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_2^4$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)$$

$$\left(\frac{64}{8} - 8\right) - \left(\frac{8}{3} - 2\right)$$

$$= -\frac{2}{3}\ sqr\ units$$

Exercise 5

1. Find the area of the region bounded by the straight lines.
   $y = 2x, x = 2$ and $x = 5.$
2. Calculate the area of the region bounded by the curve
   $y = x^3 - x^2 - 2x$, the y axis and $x = 2.$
3. Find the area enclosed by the curve $y = x^2 - 2x + 6$ with the x axis between $x = 1$ and $x = 5.$
4. Sketch the curve $y = x^2 - 4$. Find the area of the segment cut off by the x axis.
5. Calculate the areas enclosed by the following curves and the x axis.
   a) $y = x^3 - 3x^2 + 2x$
   b) $y = x^3 - 3x + 2$
   c) $y = x^3 - 4x^2 + 4x$
   d) $y = 3x^2 - x^3$

South Sudan

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