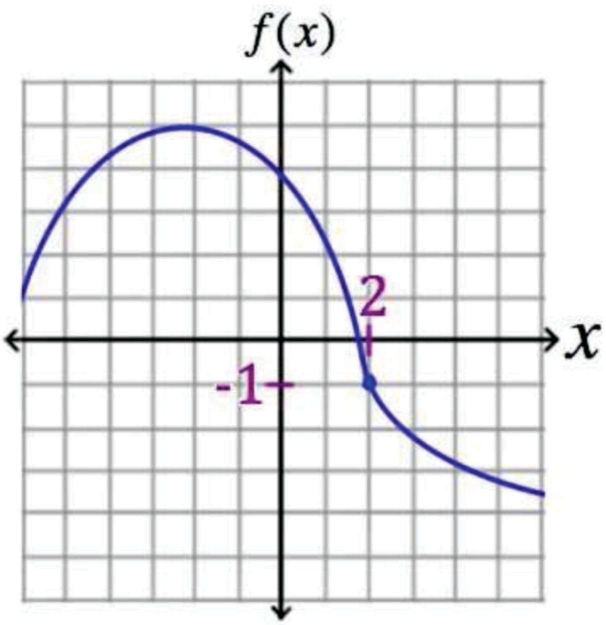
South Sudan

Secondary Additional Mathematics 4

Student's Book

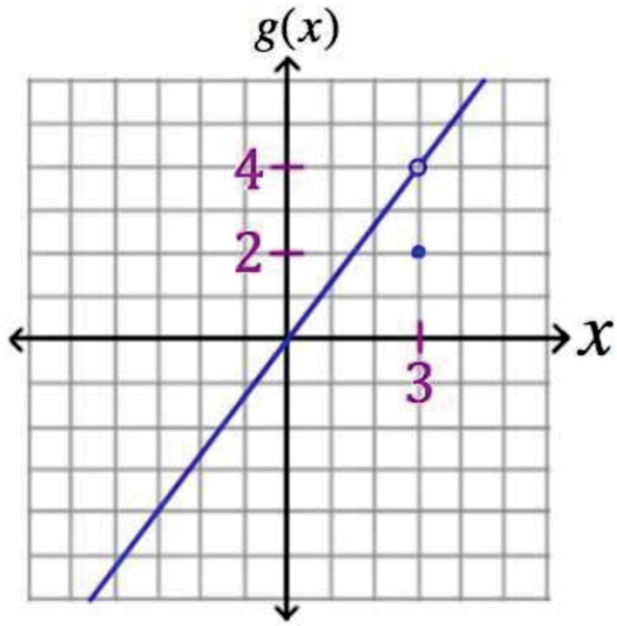
Example 3

$$\lim_{x \to 2} f(x) = -1$$



Example 4

$$\lim_{x \to 3} g(x) = 4$$



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South Sudan

SECONDARY
4

Additional Mathematics

Student's Book 4

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UNIT 1

LIMITS (rational functions)

Introduction

The knowledge of limits is very important in mathematics. It is used in differentiation calculus, integration and determining tangents. One of the basic use is determination of maximum and minimum values of variables related by two unknown for instance in maximizing area of the rectangle from a given perimeter.

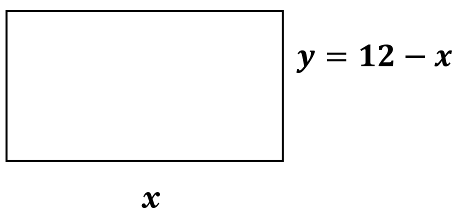
Example

What dimensions should make a rectangle of maximum area using a wire 24m.

Solution

We need to have an expression of x against one dimension expressed as the other.

Consider



Rectangle in which,

x= *length*

y= *width*

$2(x + y) = 24$ (Perimeter)

$$x + y = 12$$

$$x = 12 - y$$

The expression for area is hence

$A = lw \qquad A = x(12 - x)$

$A = xy \qquad A = 12y - y^2$

Instead of differentiation we can develop a table of w against area A when the width tends to a certain value of width.

Activity 1

i. In groups, complete the table below on different lengths and areas of a rectangle that would be formed by a wire 24m.

Width (y)	5	5 ½	5.9	6	6.1	6.5	7.0
Length (x)							
Area A							

ii. What area does the rectangle tend to have when the width tends to 6m.

iii. Write an expression for the limit of area as width tends to be 6m.

Solution.

Width (y)	5	5 ½	5.9	6	6.1	6.5	7.0
Length (x)	7	6 ½	6.1	6	5.9	5.5	5.0
Area A	35	35 ¾	35.99	36	35.99	35.75	35

ii. You notice as width tends to 6 the area tends to 36.

iv. This is expressed in limit notation as

$$\lim_{y \to 6} Area = 36$$

$$\lim_{y \to 6} (12y - y^2) = 36$$

Definition of limit

If a function $f(x)$ becomes closer and closer to a number L as x approaches a constant c from either side, then the limit $f(x)$ as x tends to c is L.

This relation-ship is written in symbol as

$$\lim_{Y \to 6} f(x) = L$$

Methods of determining limits

There are three main methods of finding limit of a function as x tends to agiven constant. These methods include;-

1) Estimating the limits numerically.
2) Estimating the limits graphically
3) Determining limit by substitution.

Estimating limits numerically

In this method a limit of a function is estimated using a table of x against f(x) as x tends to a constant c from both side of the x-values.

Example II

Activity 2

i. In pairs, complete the table below for the function $f(x) = 3x - 3$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)				?			

ii. Use the table to estimate the limit of the function $f(x) = 3x - 3$ as x tends to 2.

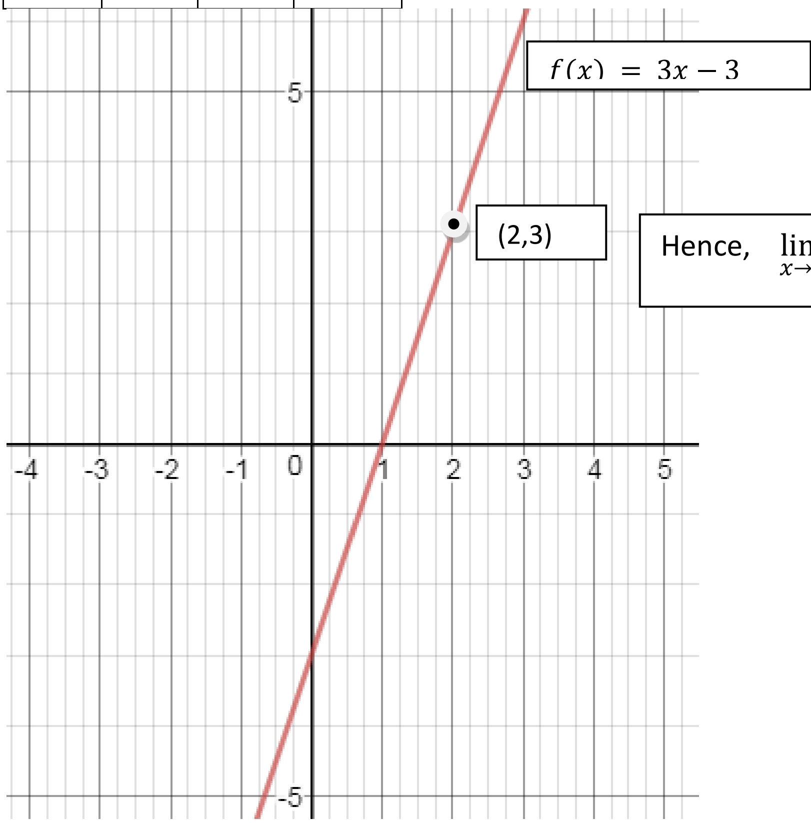
iii. Draw the graph of the function $f(x) = 3x - 3$ using the range $0 \leq x \geq 3$

Solution

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	2.700	2.97	2.997	?	3.003	3.030	3.30

$$\lim_{x \to 2}(3x - 3) = 3$$

x	0	1	3
y	-3	0	6



Hence, $\lim_{x \to 2}(3x - 3) = 3$

On drawing the graph of $f(x) = 3x - 3$ you notice the graph is straight line that is continuous since it is defined on the real number line, R

Limits of non-continuous functions

Example

Activity 3

i. In groups, complete the table below for the function $f(x) = \frac{x}{\sqrt{x+1}-1}$

X	-0.01	-0.001	-0.001	0	0.0001	0.001	0.01
$f(x)=\frac{x}{\sqrt{x+1}-1}$				?			

ii. From the table find the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$$

iii. Draw the graph of $f(x) = \frac{x}{\sqrt{x+1}-1}$ using the interval $-1 \le x \ge 4$

Solution

X	-0.01	-0.001	-0.001	0	0.0001	0.001	0.01
$f(x)=\frac{x}{\sqrt{x+1}-1}$	1.99499	1.99949	1.9999	?	2.005	2.005	2.0049

$$\lim_{x\to 0}\left(\frac{x}{\sqrt{x+1}-1}\right)=2$$

You notice as x tends to 0, the value of $\left(f(x) = \frac{x}{\sqrt{x+1}-1}\right)$ from both side tends to 2. However it is important to note the value of f (0) does not exist since $\frac{x}{0}$ is undefined. If we draw such a graph we need to show this value does not exist. Unshaded circle show the non-existence. ***Figure 1.1 below shows the graph of this function***.

Figure 1.1

Exercise 1.1

Work in groups of four students.

Determine the limit of the following at the described positions numerically.

1. $\lim\limits_{x \to -1} \quad (x^2 + x - 8)$
2. $\lim\limits_{x \to -1} \frac{(x^2+x-6)}{x+3}$
3. $\lim\limits_{x \to -1} \frac{(x+1)}{x^2-x-2}$
4. $\lim\limits_{x \to 1} \quad (5x + 8)$
5. $\lim\limits_{x \to 0} \quad 2x^2 + 3x + 3$

6. $\lim_{x\to 2} \frac{x^2-4}{x-2}$

7. $\lim_{x\to 1} \quad 6x$

8. $\lim_{x\to 4} \sqrt{x}$

9. $\lim_{x\to 3} \quad (x+4)^2$

Estimating the limits graphically

The limits can also be obtained by drawing graph of function. If x approaches a constant c from both right hand side (RHS) and left hand side (LHS) the value of the function approaches a constant L then L is the limit of such a function at c. if $f(c)$ is not defined one can check whether $f(c)$ approaches a same constant L from both RHS and LHS.

If $f(c) = l$ from LHS and RHS of a function then the $\lim_{x\to c} f(x) = l$.

Activity 6

i. In pairs, examine the graph in figure 1.2 below.
ii. From the graph determine and illustrate limit $\lim_{x\to c} f(x)$

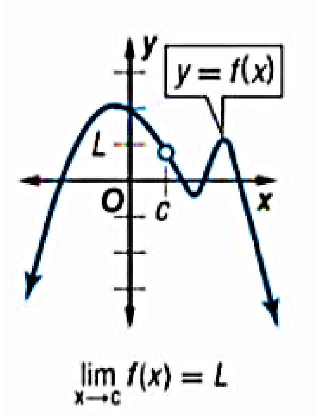


Figure 1.2

Example

a. In groups of three students study the figure 1.3 below that shows a graph of a function f(x)

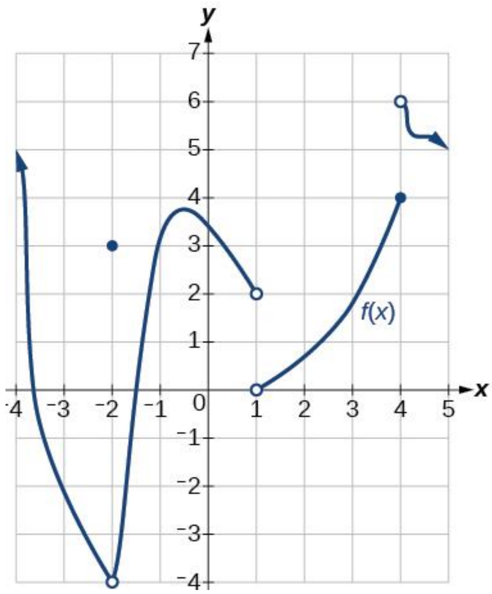


Figure 1.3

b. Using the graph determine the limits below.

i. $\lim_{x \to -2} f(x)$

ii. $\lim_{x \to 1} f(x)$

iii. $\lim_{x \to 4} f(x)$

c. In each limits in (b) above give reason for your answer.

Solution

a) $\lim_{x \to -2} f(x) = -4$ the function approach -4 from both sides

b) $\lim_{x \to 1} f(x)$

Does not exist since the value of $f(x)$ approaches two values, on L.H.S approaches 2 and that from R.H.S approaches 1. They should approach the same figure.

c) $\lim_{x \to 4} f(x)$

Does not exist, the function approaches two values , 4 and 6, from both sides of the functions.

Example 2

a. Sketch the graph of a function $f(x) = \frac{x^3+8}{x+2}$ for $-4 \leq x \leq 4$

b. ***Find*** $\lim_{x \to 2} \frac{(x^3+8)}{(x+2)}$

Solution

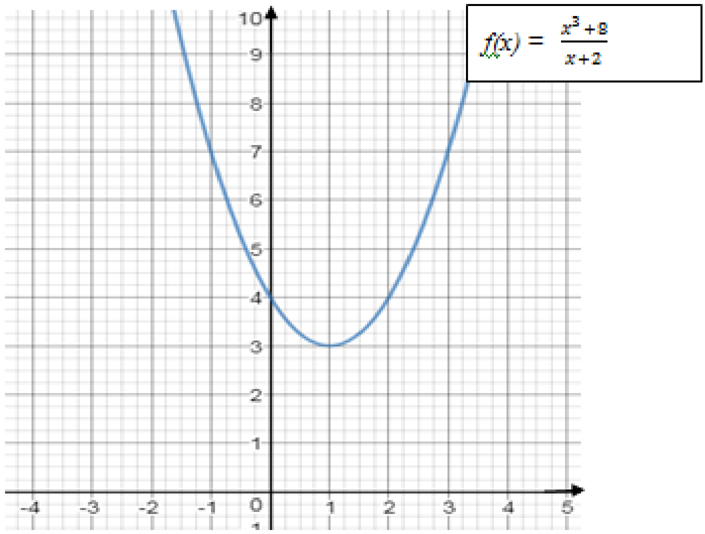
X	-4	-3	-2	-1	0	1	2	3	4
Y	28	+19	∞	7	4	3	4	7	12

Since $f(-2)$ is not defined we need to critically search the tendency of f(x) as approaches -2 from R.H.S and L.H.S.

Using table below.

X	-2.5	-2.3	-2.2	-2.1	2	-1.95	-1.9	-1.8	-1.7	-1.5
Y	15.25	13.89	13.24	12.61	∞	11.7	11.41	10.84	10.29	9.25

You notice as x approaches c (-2) from both R.H.S and L.H.S the value $f(x)$ approaches 1 as illustrated in figure 1.4 below.



Hence, $\lim_{x \to -2} \frac{(x^3+8)}{(x+2)} = 12$

Graphs of functions defined by two expressions.

Example 1

i. Graphically illustrate the graph of f(x) on the interval $-3 \leq x \geq 3$ if:

$$f(x) = \{\begin{matrix} 1 & for x \geq 0 \\ 0 & for x \leq 0 \end{matrix}$$

ii. Using the graph determine the value of:

a. $\lim_{x \to 2} f(x) = \{\begin{matrix} 1 & for x \geq 0 \\ 0 & for x \leq 0 \end{matrix}$

b. $\lim_{x \to 0} f(x) = \{\begin{matrix} 1 & for x \geq 0 \\ 0 & for x \leq 0 \end{matrix}$

Solution

a)

X	-3	-2	-1	0	1	2	3
Y	0	0	0	0 or +1	1	1	1

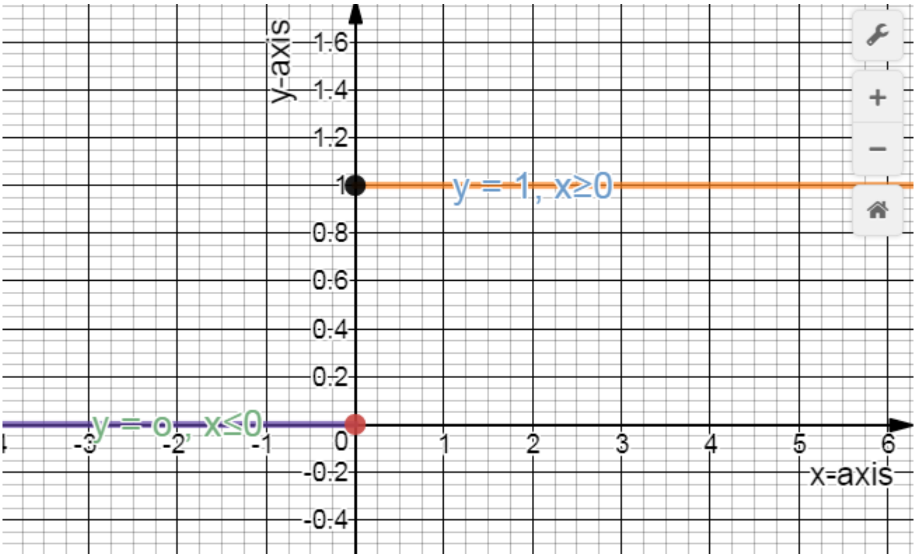


Figure 1.5

a. $\lim_{x \to 2} f(x) = \{ \begin{matrix} 1 & for x \geq 0 \\ 0 & for x \leq 0 \end{matrix} = 1$

b. $\lim_{x \to 0} f(x)$ Does not exist since f(o) has two solutions. Do not approach to same constant.

Example 2

a. In pairs, draw the graph of,

$$f(x) = \begin{cases} x + 1 & for \quad x < 1 \\ -\frac{1}{2}x + 4\frac{1}{2} & for \quad x > 0 \\ 3 & \text{for x = 1} \end{cases}$$

b. ***Using the graph find***

i. $\lim_{x \to 0} f(x)$

ii. $\lim_{x \to 1} f(x)$

Solution

1st expression (x = -1, 0)   2nd expression (x = 3)

x	-1	0	1	3
y	0	1		3

1 or 4

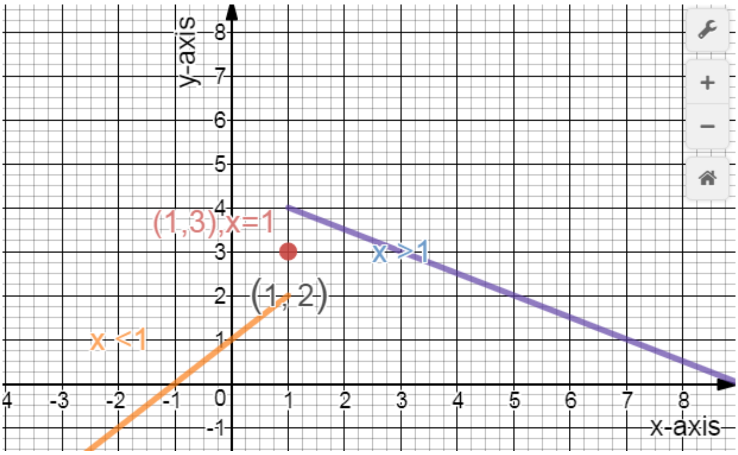


Figure 1.6

i. $\lim_{x\to 0} f(x) = 1$

ii. $\lim_{x\to 1} f(x)$ *does not exist since the solutions are more than one*

Limits that exist when a function is not defined.

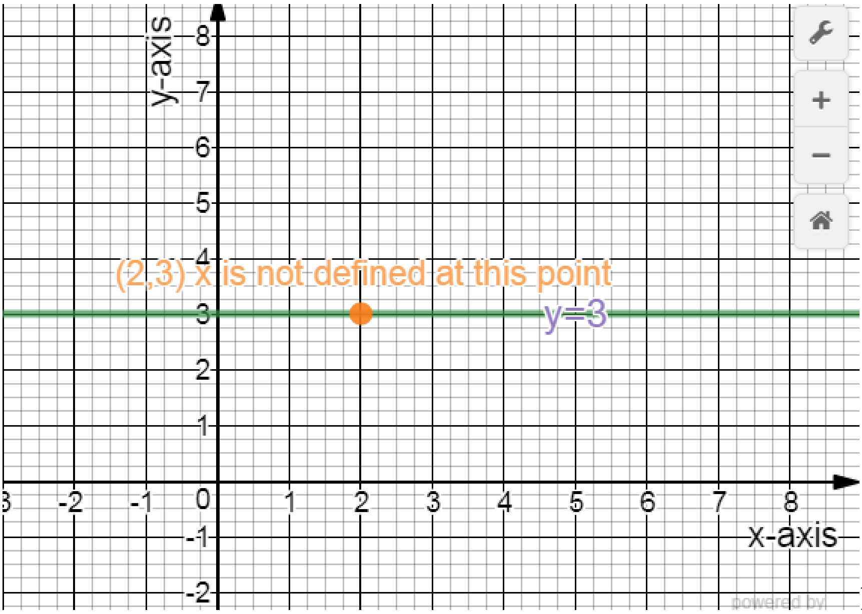
Example

a) In pairs draw a graph of $f(x) = \begin{cases} 3 & for\ -\infty \le x \le +\infty \\ & but \quad x \ne 2 \end{cases}$

b) Find $\lim_{x\to 2} f(x)$

Solution

The graph of the f(x) is



$\lim_{x \to 2} f(x) = 3$

Unbounded limits

If the value of $f(x)$ is not defined then the limit does not exist. The undefined solution is as a result of $\frac{k}{0}$ where k is a real number on an expression. On the graph there points are illustrated as asymptote.

Example

a) Draw the graph of $f(x) = \frac{1}{x^2}$ for $-3 \leq x \geq 3$

b) Using the graph determine the $\lim_{x \to 2} f(x)\, for f(x) = \frac{1}{x^2}$

Solution

x	-3	-2	-1	0	1	2	3
y	0.111	0.25	1	∞	1	0.25	0.111

As x approaches o from R.H.S and become extremely big hence the y-axis is an asymptote as shown in figure 1.7 below.

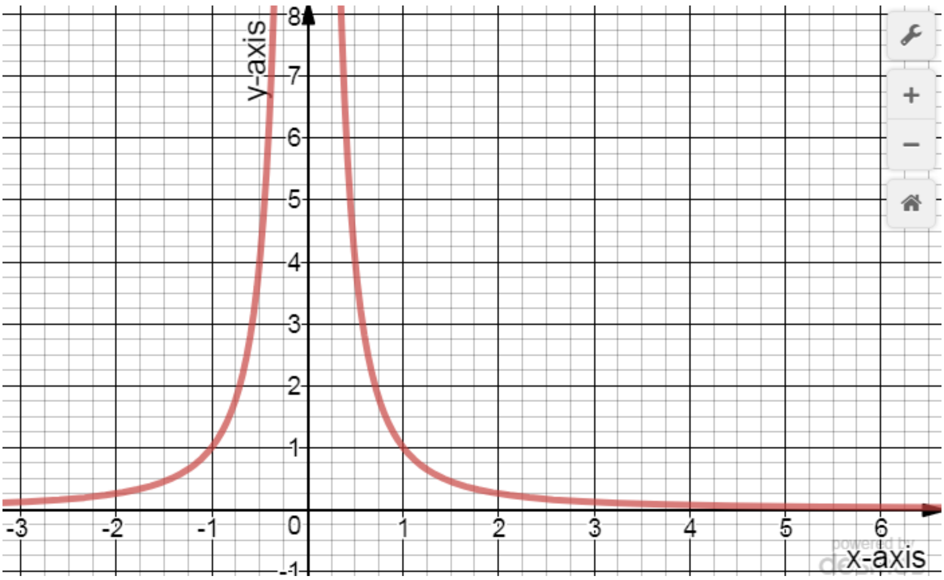


Figure 2.7

You notice the value $f(x)$ does not approach a real number L as x approaches o hence the limit does not exist. The solution of $f(c)$ is said to be unbounded.

Oscillating behavior of a curve

If a curve of a function $f(x)$ oscillates as x tends to c then the limit cannot exist at $x = c$ since the solution of $f(x)$ does not tend to one real number L from L.H.S and R.H.S of a function.

Example

a) Sketch the curve of $f(x) = \sin\left(\frac{1}{x}\right)$

b) Find $\lim\limits_{x\to 0} sin\frac{1}{x}$

Solution

The table that guides us to sketch is:

x	-3	-2	-1	0	1	2	3
$f(x)$	0.35	-0.47	-0.84	∞	0.8	0.47	0.33

At x=0 the f(x) is not defined to investigate is as x tend to 0 the f(x) approaches a constant we find that the f(x) oscillates about 1 and –u.

-0.5	-0.99	0.999	0	0.999	0.99	0.5	0.25	0.05
-.909	-.846	-8420	∞	0.842	0.846	0.9	-0.75	0.91

The figure 1.8 below shows how the $f(x) = \sin\frac{1}{x}$ oscillates.

In general, the limit of a function f(x) does not exist as $x \to c$ if

1. $f(x)$ approaches different values from right hand side of $x = c$ and left hand side of $x = c$.
2. $f(x)$ increases or decreases without bound as x approaches c. The value as the value of $f(c) = \infty$ (is undefined).
3. $f(x)$ oscillates between and fixed values as the value of x approaches c.

Exercise 1.2

Work in groups.

1. By use of graph estimate the limits:

    a) $\lim\limits_{x\to 1} \frac{x^3+x-x^2-1}{x-1}$

    b) $\lim\limits_{x\to 3} f(x) where f(x) = \begin{cases} 2 & x \neq 3 \\ 0 & x = 3 \end{cases}$

c) $\lim_{x\to 0} \frac{(x)}{x}$

d) $\lim_{x\to 0} \frac{1}{x^3}$

e) $\lim_{x\to 1} \frac{x^3-x}{x-1}$

f) $\lim_{x\to 0} cos\,(\frac{1}{x})$

2. Find the value of the following limits by drawing the graphs of the following functions.
   a) $\lim_{x\to 2} x^2$
   b) $\lim_{x\to 1} f(x) where f(c) = 2x + 3$
   c) $\lim_{x\to 1} f(x) where f(x) = \begin{cases} 0 & x \leq 4 \\ 2x & 4 \leq x \leq 6 \\ 4x^2 & x \geq 6 \end{cases}$
3. Given the f(x) is

$$f(x) = \begin{pmatrix} x^2 + 1 & -x \leq 2 \\ x & 2 \leq x \leq 3 \\ 4 & 3 \leq x \leq 6 \end{pmatrix}$$

$4x + 2 + x^2$ x≥6

Draw a the graph of 0≤x≤8

Find the value of;-

a) $\lim_{x\to 0} f(x)$

b) $\lim_{x\to 2} f(x)$

c) $\lim_{x\to 5} f(x)$

d) $\lim_{x\to 6} ff(x)$

e) $\lim_{x \to 7} f(x)$

In each of the following questions draw the graph of the functions involved. For each function draw its graph, and state if the limit, if it exist, if it does not exist explain why.

4. $\lim_{x \to 0} \quad 12 - x^2$

5. $\lim_{x \to -2} \left(\frac{X^2-4}{X+2}\right)$

6. $\lim_{x \to -2} \left(\frac{[X+3]}{X+3}\right)$

7. $\lim_{x \to 1} sin\left(\frac{ux}{2}\right)$

8. $\lim_{x \to 1} \left(\frac{1}{X-1}\right)$

9. $\lim_{x \to -\frac{\pi}{2}} \tan x$

10. $\lim_{x \to 0} \ 2\cos\left(\frac{\pi}{x}\right)$

11. $\lim_{x \to \frac{\pi}{x}} \quad \sec x$

12. $\lim_{x \to 2} f(x) where f(x) \begin{cases} 2x+1 & x < 2 \\ x+3 & x \geq 2 \end{cases}$

13. $\lim_{x \to 2} f(x) where f(x) \begin{cases} -2x x \leq 2 \\ x^2 - 4x + 1 \quad x \geq 2 \end{cases}$

14. $\lim_{x \to 2} \ 5x + 2$

15. $\lim_{x \to 1} (2x^2 + x - 4)$

Determining limits by direct substitution

Apart from use of numerical or graphing method to determine limits of e function f as x approaches a constant c can also be obtained by substituting $x = c$ on the function $f(x)$.

The following theorems that can be obtained numerically or from graph are used.

	Theorem	Explanation
	$\lim_{x \to 2} c = c$	The limit of a constant (c) is the same constant (c). This s because the value of x does not affect a variable.
	$\lim_{x \to a} x = a$	Limit of x as x approaches a constant a is a.
	$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$	The limit of a function multiplied by a constant is the same as constant multiply by its limit.
	$\lim_{x \to c} (f(x) + g(x) + h(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) + \lim_{x \to c} h(x)$	Limit of a polynomial is the same as sum or difference limit of each term of the polynomial.
	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$	The limit of a function is the same as the limits of numerator divided by limit of denominator

	$\lim_{x \to c} f(x).g(x)$ $= \lim_{x \to c} f(x). \lim_{x \to c} g(x)$	Limit of a product is the product of the limits.
	$\lim_{x \to c} x^n = \left(\lim_{x \to c} x\right)^n$	The limit of a power to a function is the power to the limit
	$\lim_{x \to c} x^3 \quad \sqrt[n]{x} = \sqrt[n]{c}$	Limit of a root is the root of the substitute made.

Example 1

Find $\lim_{x \to 4} 4$

Solution

$$\lim_{x \to 4} 4 = 4$$

Example 2

The monthly salary of an employee remain the same, is SSP10 000 over a whole year. Write an expression of salary against time (t). Determine the limit of salary as the time approaches second month.

Solution

$$s(t) = k$$

$$s(t) = 10\,000$$

$$\lim_{x \to 2} 10\,000 = 10\,000$$

This is because salary is constant function of t.

Example 3

Find $\lim_{x \to 3} x$

Solution

$$\lim_{x \to 3} x = 3$$

Example 4

Find $\lim_{x \to 3} (x + 2)$

Solution

$$\lim_{x \to 3} (x + 2) = 3 + 2 = 5$$

Example 5

Find, $\lim_{x \to 4} 3\mathrm{x}$

Solution

$$\lim_{x \to 4} 3\times 4 = 12$$

Or

$$\lim_{x \to 4} 3\mathrm{x} = 3\lim_{x \to 4} \mathrm{x}$$

$$= 3\times 4$$

$$=12$$

Example 6

Find $\lim_{x\to 1} 2x + 3$

Solution

$$\lim_{x\to 1}(2x+3) \quad = \lim_{x\to 1} + \lim_{x\to 1} 3$$

$$= 2+3 = \ 5$$

Example 7

Find

$$\lim_{x\to -2}\frac{3x}{4}$$

Solution

$$\lim_{x\to -2}\frac{3x}{4} = \frac{\lim_{x\to -2} 3x}{\lim_{x\to -2} 4}$$

$$= \frac{\lim_{x\to -2} x}{4} \ = \frac{3(-2)}{4} \ = \ -{}^{3}/_{2}$$

Example 8

$$\lim_{x\to 2} x^2$$

Solution

$$\lim_{x\to 2} x.x \quad = \ \lim_{x\to 2} x \ .\lim_{x\to 2} x$$

$$= 2\times 2 = 4$$

Example 9

Find,

$$\lim_{x\to 2} x^3$$

Solution

$$\lim_{x \to 2} x^3 = \left(\lim_{x \to 2} x\right)^3 = 2^3 = 8$$

Alternatively

One can substitute directly to the function.

$$\lim_{x \to 2} x^3 = 2^3 = 8$$

Trigonometric limits

	Trigonometric limits
a.	$\lim_{x \to \pi} \sin x = \sin \pi = 0$
b.	$\lim_{x \to \pi} \cos x = \cos \pi = 1$
c.	$\lim_{x \to \pi} \tan x = \lim_{x \to \pi} \frac{\sin x}{\cos x} = \frac{\lim_{x \to \pi} \sin x}{\lim_{x \to \pi} \cos x} = \frac{0}{1} = 0$

In general most limits for x ending to c can be solved by substituting x=c to the function.

Example

In groups, discuss and find the solution to the following limits.

a) $\lim_{x \to 4} 4x$

b) $\lim_{x \to \pi} \frac{\tan x}{x^2}$

c) $\lim_{x \to \pi} (x^2+1) \cos x$

Solution

a) $\lim_{x \to 4} 4x = 4 \times 4 = 16$

b) $\lim_{x \to \pi} \frac{\tan x}{x^2} = \frac{\tan \pi}{\pi^2} = \frac{0}{\pi^2} = 0$

c) $\lim_{x \to \pi} (x^2 + 1) \cos x = \lim_{x \to \pi} (x^2 + 1) \,.\, \lim_{x \to \pi} \cos$

a. $= (\pi^2+1).1$

$= \pi^2+1$

Exercise 1.3

In pairs, find the solution to the following limits using any suitable method

1. $\lim_{x \to 3} (x+4)^2$
2. $\lim_{x \to 1} x^2 + 2x + 3$
3. $\lim_{x \to 2} 2x^2 + 4$
4. $\lim_{x \to 1} \frac{x^2+3x+4}{x}$
5. $\lim_{x \to 5} (10-x^2)$
6. $\lim_{x \to 1} \frac{1}{2} x^3+x^2+5$
7. $\lim_{x \to 3} \frac{5x+3}{x^2 1}$
8. $\lim_{x \to \pi} 3sinx$
9. $\lim_{x \to \pi} (\tan x + \cos x)$
10. $\lim_{x \to 3} \sqrt{x^2} + n$
11. $\lim_{x \to 2} \frac{x-2}{x^2+2x+1}$
12. $\lim_{x \to 4} x^x$
13. $\lim_{x \to 2} 4^x$
14. $\lim_{x \to 1} 7^{(x+1)}$
15. $\lim_{x \to 2} \sqrt[3]{x^2+2}$
16. $\lim_{x \to 0} \frac{2x^2+4}{x+1}$
17. $\lim_{x \to 1} \frac{x^4+16}{x^2+1}$
18. $\lim_{x \to 3} e^{2x}$
19. If $f(x) = \frac{2x^2+x}{x^2-4}$, find $\lim_{x \to 1} f(x)$

Determining limits by indirect substitution

The method of direct substitution is not always applicable. It does not work for some functions. Such substitutions results include;

i. $\frac{0}{0}$ zero divide by zero since it is undefined
ii. $\frac{\infty}{\infty}$ infinity divide by infinity since it is also undefined
iii. $\frac{x}{0}$a number divide by zero since its undefined and give an error.

Note:$\frac{0}{x}$ =0 hence such simplification is soluble at x.

If the substitution brings out the above scenarios the techniques used includes;

1. Simplification by dividing by common factor.
2. Rationalizing numerator or denominator.
3. Falling back to numerical or graphing method.
4. Dividing by x^n where n is the highest power for expressions that give$\frac{\infty}{\infty}$.

Determining limits by simplification using the common factor.

Most algebraic fractions can be simplified to yield definite limits which are not of the nature of $\frac{0}{0}$, $\frac{x}{0}$ and $\frac{\infty}{\infty}$.

These simplification can also be done using expansion identifies.

Examples

1. In pairs, expand the following expression.
   i. $(x+y)^2$
   ii. $(x-y)^2$
   iii. $(x+y)(x-y)$
   iv. $(x-y)(x^2+y^2+xy)$
2. Hence, simplify $\frac{x^4-16}{x-2}$

Solution

1. $(x + y)^2 = x^2 - 2xy + y^2$

$(x – y)^2 = x^2 - 2xy + y^2$

$(x^2 – y^2)) = (x + y)(x – y)$

$(x^3 - y^3)= (x – y) (x^2 + y^2 + xy)$

2. $x^4 - 16$, is of the form , $a^2 - b^2 = (a + b)(a - b)$, applying this

$x^4 - 16 = (x^2)^2 - 4^2 = (x^2 - 4)(x^2 + 4) = (x^2 + 4)(x + 2)(x - 2)$

Hence $\frac{x^4-16}{x-2} = \frac{(x^2+4)(x+2)(x-2)}{(x-2)} = (x^2 + 4)(x + 2)$

Example 1 (in groups of three students)

a. Simplify the expression $\frac{x^2-4x+4}{\text{x}-2}$ into a single fraction.
b. Determine the

$$\lim_{x \to 2} \frac{x^2 - 4x + 4}{\text{x} - 2}$$

Solution

a. x^2-$4x$+4= x^2-$2x$-$2x$+4

=$x(x$-$2)$-$2(x$-$2)$

=$(x$-$2)$ $(x$-$2)$ *hence*

$$\frac{x^2 - 4x + 4}{\text{x} - 2} = \frac{(x - 2)\ (x - 2)}{(\text{x} - 2)} = (\text{x} - 2)$$

b.

$$\lim_{x \to 2} \frac{x^2-4x+4}{\text{x}-2} = \lim_{x \to 2} (x - 2) = 2 - 2 = 0$$

Note. By direct substitution to the expression

$$\lim_{x \to 2} \frac{x^2 - 4x + 4}{\text{x} - 2}$$

The limit is not defined since

$$\lim_{x \to 2} \left(\frac{x^2-4x+4}{x-2}\right) = \frac{2^2+4(-2)+4}{2-2} = \frac{0}{0} \quad \text{this is undefined}$$

This limit is said to exist but the function is not defined at x = 2. Such limits are determined only after simplification of their expressions.

Example 2

a. In groups, simplify the expression $\frac{x^4-16}{x-2}$

b. Determine the $\lim_{x \to 2} \frac{x^4-16}{x-2}$

Solution

By using $(a^2-b^2) = (a+b)\,(a-b)$

$x^4-16 = (x^2)^2-(4)^2 = (x^2+4)\,(x^2-4) = (x^2+4)\,(x+2)\,(x-2)$

Hence $\frac{x^4-16}{x-2} = \frac{(x^2+4)\,(x+2)(x-2)}{(x-2)} = (x^2+4)\,(x+2)$

$$\lim_{x \to 2} \frac{x^4-16}{x-2} = \lim_{x \to 2} (x^2+4)\,(x+2) = (2^2+4)\,(2+2) = 32$$

Example 3

In groups, find the value of $\lim_{x \to 3} \frac{x^3-27}{x-3}$

Solution

By direct substitution we get un defined form of $\frac{0}{0}$ hence we make substitution.

Using $a^3-b^3 = (a - b)\,(a^2 + b^2 + 2ab)$

$x^3 - 27 = \quad x^3 - 3^3$

$= (x-3)\ (x^2+9+6x)$

$= (x-3)\ (x^2+6x+9)$

Hence

$$\lim_{x\to 2}\left(\frac{x^3-27}{x-3}\right) = \lim_{x\to 3}\frac{(x-3)(x^2+6x+9)}{(x-3)}$$

$$= \lim_{x\to 3}\ (x^2+6x+9)$$

$$= 3^2 + 6 \times 3 + 9$$

$$= 27$$

Exercise 1.4

Work in groups. Find the value of the following limits.

1. $\lim\limits_{x\to 3}\frac{x^2-x-16}{x+3}$
2. $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$
3. $\lim\limits_{x\to 1}\frac{(x-1)^2}{(x-1)}$
4. $\lim\limits_{x\to 1}\frac{-x-16}{-1+x+x^3}$
5. $\lim\limits_{x\to -4}\frac{16-x^2}{4+x}$
6. $\lim\limits_{x\to 1}\frac{x-1}{x^2+x-2}$
7. $\lim\limits_{x\to 5}\frac{x^2-10x-25}{x^2-4x-5}$
8. $\lim\limits_{x\to 5}\frac{x^2-10x-25}{x-5}$
9. $\lim\limits_{x\to 2}\frac{x^3-8}{x^2-4}$
10. $\lim\limits_{x\to 0}\frac{x^2+2x-1}{x^3-x}$

Determining limit by rationalizing part of the expression

If a direct substitution to a function leads to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{x}{0}$ or then rationalizing can eliminate such scenario and results of limit obtained.

Example 1

In pairs, find

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$$

Solution

Note direct substitution gives results of $\frac{\sqrt{2}}{0}$ hence limit cannot be defined by this expression.

Rationalizing numerator

$$=\frac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x \quad (\sqrt{x+9}+3)}$$

$$=\frac{x+9-9}{x\,(\sqrt{x+9}+3)}$$

$$=\frac{1}{(\sqrt{x+9}+3)}$$

$$=\lim_{x\to 0}\frac{\sqrt{x+9+3}}{x}==\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}$$

$$=\frac{1}{\sqrt{9}+3}$$

$$=\frac{1}{3+3}$$

$$=\frac{1}{6}$$

Example 2

Evaluate:

$$\lim_{x\to 0}(x+2)^{1/x}$$

Solution

Note *1/x for x=0=1/0* which is not defined hence f(x) has no defined value at x=0. However we can investigate whet ether at x approaches 0 from L.H.S and R.H.S the value of the function approaches given value.

Using y= $(x+2)^{1/x}$

x	-0.9	-0.09	-0.009	0	0.009	0.09	0.9
f(x)	0.8995	0.00075	0.5xa0-34		4.5x1034	3.6x104	

Undefined

You notice the limit approach different values on other sides of 0 hence the limit does not exist.

You notice that only one point of left hand side L.H.S and another on right hand side R.H.S of c for

$\lim_{x\to c} f(x)$can be used to predict the limit.

Example 3

Find the, $\lim_{x\to 0} \frac{sinx}{x}$

Solution

Direct substitution give undefined value of $\frac{0}{0}$ let us consider the graph of f(x) = $\frac{\sin x}{x}$ and one value on L.H.S and R.H.S close to 0.

Table

x	-0.001	0	+0.001
$\sin x$	-0.000999	-	0.000999
$f(x) = \frac{\sin x}{x}$	-0.9999	-	0.9999

You notice as x $\to$ 0 the value on L.H.S approach 1 and those on R.H.S also hence

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

The limit can be clearly shown by the graph of $y = \frac{\sin x}{x}$ below.

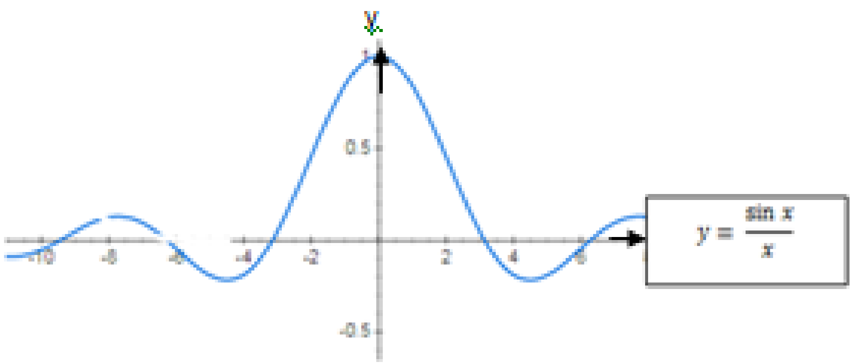


Figure 1.9

Exercise 1.5

Work individually.

Using any suitable method determines the limit of the following;

1. $\lim_{x\to 1}\frac{2x^2+3x}{x}$
2. $\lim_{x\to 0}2\frac{\sin x}{x}$
3. $\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}$
4. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2}$
5. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$
6. $\lim_{x\to 0}\frac{3\,sinx}{5x}$
7. $\lim_{x\to 0}\frac{3\,sinx}{x}$

One side limit

It has been illustrated using graphical method that a limit L of a f(x) exist if as x approaches c the graph approaches l from both side of the function.

$\lim_{x\to c} f(x) = l$

However the limit can approach a certain value L, on one side and a different value L_2 on the other side such limits are called **one-side-limit**. As we approach c from **left hand** side the limit abbreviated by the symbol c^- and as we approach c from right hand side we express it as c^+.

Hence

Limit from left is written as $\lim_{x\to c^-} f(x) = L_1$

Limit from right is written as $\lim_{x\to c^+} f(x) = L_2$

Note:

The existence of a limit on L.H.S and R.H.S that are different L_1 and L_2 implies that the limit of the function does not exist. A function only exist of $L_1=L_2$.

Example

Evaluate a) $\lim\limits_{x \to 0^-} \frac{|2x|}{x}$

b) $\lim\limits_{x \to 0^+} \frac{|2x|}{x}$

Solution

By numerical method. On LHS and RHS

x	-3	-2	-1	1	1	2	3
$\|2x\|$	6	4	2	0	2	4	6
$fx = \frac{\|2x\|}{\|x\|}$	-2	-2	-2		1	1	1

Undefined

The graph of f(x) is shown in figure 1.10 below

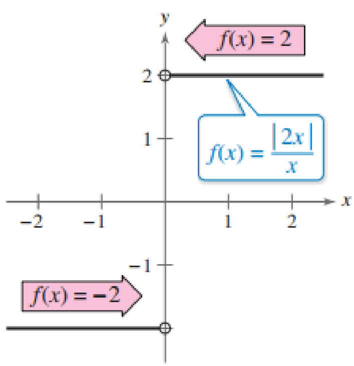


Figure 1.10

From the graph you notice there exist a limit on left hand side and right hand side.

$\lim_{x \to 0^-} \frac{|2x|}{x} = -2$

$\lim_{x \to 0^+} \frac{|2x|}{x} = -2$

$\lim_{x \to 0} \frac{|2x|}{x}$ does not exist since the function has two solutions, -2 and 2.

Exercise 1.6

Work in groups.

1. Given the function f(x) as,

$$f(x) = \begin{cases} 4 - x & , x < 1 \\ 4x - x^2 & , x > 1 \end{cases}$$

a. Draw the graph of the function $f(x)$.

b. Determine

i) $\lim_{x \to 1^-} f(x)$

ii) $\lim_{x \to 1^+} f(x)$

iii) $\lim_{x \to 1} f(x)$

2. To ship a cargo on a port the cost first tone is charged SSD.1,780, the weight between first ton and second tone SSD.1,920 and the rest SSD.2,060. Using x to represent the weight of the package attempt the following questions below.

   a. Form a function f(x)
   b. Illustrate these function on a graph.
   c. Find;

i) $\lim_{x \to 2^-} f(x)$

ii) $\lim_{x \to 2^+} f(x)$

iii) $\lim_{x \to 2} f(x)$

3. In figure 11.1 below shows a graph of the change in mass in grams of withering flower over the first four days after it is cut from a stem.

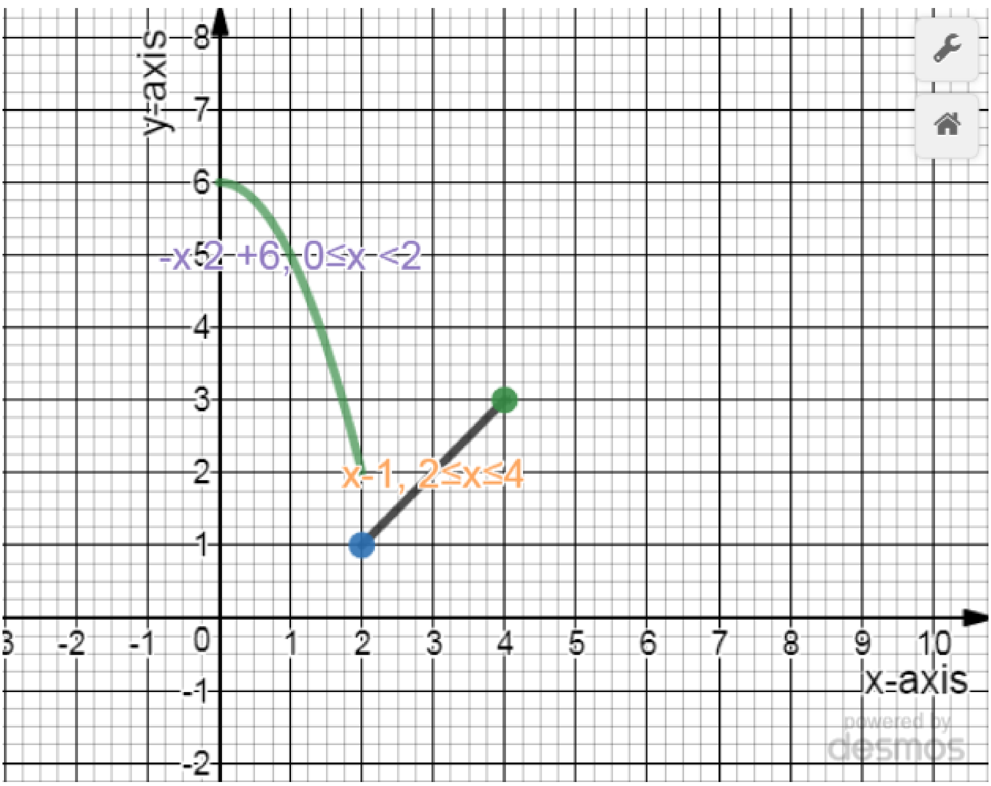


Figure 1.11

Using the graph determine

i) $\lim_{x \to 2^+} f(x)$

ii) $\lim_{x \to 2^-} f(x)$

iii) $\lim_{x \to 2} f(x)$

iv) $\lim_{x \to 3} f(x)$

4. If $f(x) = \begin{cases} x^2 - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

Find,

i) $\lim_{x \to 2^+} \begin{cases} x^2 - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

ii) $\lim_{x \to 2^-} \begin{cases} x^2 - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

iii) $\lim_{x \to 2} \begin{cases} x^2 - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

iv) $\lim_{x \to 3} \begin{cases} x^2 - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

5. Given that $f(x) = \frac{x^3 - 1}{x}$

Find the value of

i) $\lim_{x \to 1} f(x)$

ii) $\lim_{x \to -1} f(x)$

iii) $\lim_{x \to 1^-} f(x)$

iv) $\lim_{x \to 0} f(x)$

v) $\lim_{x \to 1^+} f(x)$

6. Show that the limit of $f(x) = \frac{x^2 - 1}{x + 1}$ exist at x=1

7. Given that $f(x) = \frac{|x|}{x}$

a) Draw the graph of f(x)

b) Find the value of;

i) $\lim\limits_{x \to 1^-} f(x)$

ii) $\lim\limits_{x \to 1^+} f(x)$

iii) $\lim\limits_{x \to 1} f(x)$

Continuity of a function

If a function is continuous then its curve has no breaks gaps or holes.

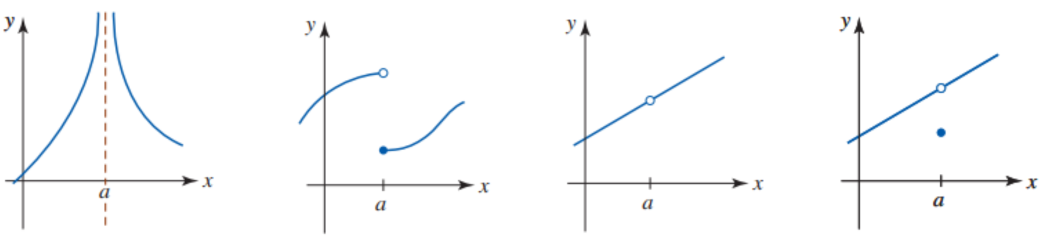
If a function f(x) is continuous at a point x=a then f(a) can be obtained by direct substation. If not then f(x) is not continuous at x=a.

For a function f(x) to be continuous a then at a function satisfy to all the three conditions below.

1. f(a) is defined - direct substitution does not give ∞, $\frac{0}{0}$
2. $\lim\limits_{x \to a} f(x)$ exist -the limit exist at a.
3. $\lim\limits_{x \to a} f(x) = f(a)$ - the value of f(a) is one by

   substitutions to the equation

Figure 1.12 a-d below show sample of non-existing continuity.



(a) $\lim\limits_{x \to a} f(x)$ does not exist and $f(a)$ is not defined

(b) $\lim\limits_{x \to a} f(x)$ does not exist but $f(a)$ is defined

(c) $\lim\limits_{x \to a} f(x)$ exists but $f(a)$ is not defined

(d) $\lim\limits_{x \to a} f(x)$ exists, $f(a)$ is defined, but $\lim\limits_{x \to a} f(x) \neq f(a)$

Figure 1.12

If a rational fraction f(x) = $\frac{g(x)}{h(x)}$ is continuous at a then h (a)$\neq$ 0. Since f (a) will not be defined by dividing by a denominator.

Example

Determine whether the function f(x), g(x) and h(x) are continuous at 1.

a) f(x) = $\frac{x^2-1}{x-3}$

b) g(x)= $\begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

c) h(x) = $\begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

Solution

a) f(x) = $\frac{x^2-1}{x-1}$

Is $f(x)$ defined? Substitute x = 1

F (1) = $\frac{0}{0}$ not defined hence not continuous.

We do not need to verify the 2nd and 3rd condition since the first has failed.

b) f(x) = $\frac{x^2-1}{x-1}$

1st is the $g(x)$ defined at g(1)? Yes from 2nd expression g(1)= 2.

2nd does $\lim\limits_{x\to 1}$ g(x) exist? $g(x) = \frac{x^2-1}{x-1}$ = (x+1)

$\lim\limits_{x\to 1}$g(x) $\{x + 1\}$ yes it exist.

3rd – is $\quad g(a) = \lim_{x \to a} \; g(a) = \lim_{x \to a} \; g(x)$

$g(1) = 2$

$\lim_{x \to 1} g(x) = 2$

Yes they are the same. The three conditions are satisfied hence g(x) is continuous at g(1).

c) $h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 3 & x = 1 \end{cases}$

1st is $h(x)$, h(1) defined? Yes

h=3

2nd does $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \; (x+1) = \lim_{x \to 1} \; x+1 = 2$

3rd if $\quad h(x) = \lim_{x \to a} \; h(x)$

$h(1)=3$

$\lim_{x \to 1} h(x) = 2$

The 3rd condition is not met hence $h(x)$ is not continuous at h(1).

Continuity on an interval

An interval is a selected range of x value under investigation for continuity of $f(x)$.

There are two types of continuity interval.

On an open interval (a, b)

If a function is continuous at open interval (a, b) and in addition $\lim_{x \to a^+}$

$f(x) = f(a)$ and $\lim_{x \to b^+}$ f(a)=f(b).

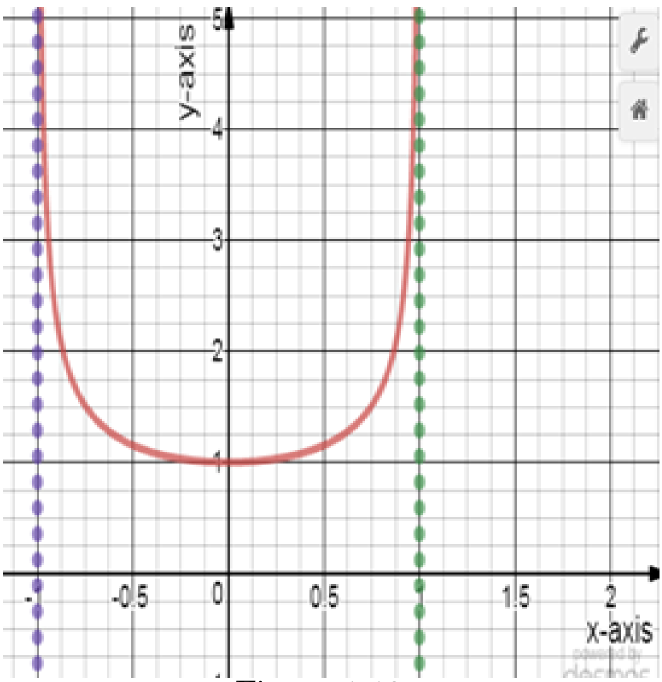
The limit to the right of a and to the left of b are defined hence the values of f(a) and f(b) are defined by continuity condition then the function is closed and one can obtain.

$$a \leq x \leq b$$

Example

In groups of four students determine

i. the asymptotes of the function and show them on the diagram.
ii. if the graph of $y = f(x) = \frac{1}{\sqrt{1-x^2}}$ shown in figure 1.13 below is continuous at;-
   b) open interval (-1,1)
   c) closed interval $[-1, 1]$



$f(x) = \frac{1}{\sqrt{1-x^2}}$

Figure 1.13

Solution

Clearly all values of open interval (-1,1) are defined.

$\lim_{x \to -1^-} f(x)$ is undefined.

$\lim_{x \to -1^+} f(x)$ is undefined.

Hence f(x) = $\frac{1}{\sqrt{1-x^2}}$ is continuous at open interval (-1, 1) only. At closed interval $[-1\,, 1]$ it is not continuous.

Example 1.7

Work in groups.

1. Determine whether f(x) = $\frac{x^3-1}{x-1}$ is continuous at 1.
2. a) Draw the graph of f(x) = $\begin{Bmatrix} \frac{x^3-1}{x-1} & , x \neq 1 \\ 2 & x = 1, \end{Bmatrix}$

   b) Determine whether the function is continuous at $x = 1$.
3. The weight of a snake for three years was noted to change piecewise according to the function below.

$$f(x) = \begin{Bmatrix} x^2 & 0 < x < 2 \\ 5 & x = 2 \\ -x + 6 & x > 2 \end{Bmatrix}$$

   a) draw the graph of the function f(x)
   b) Determine if the function is continuous at x = 2.
4. A cow was tethered next to a wall as shown in figure 1.14 below.

Figure 1.14

The furthest area that the cow would step on form the function $f(x) = \left|\sqrt{x^2 - 1}\right|$ at an interval $-1 \leq x \geq 1$ $(in\ decameters)$ Figure 1.15 below show the graph of these function.

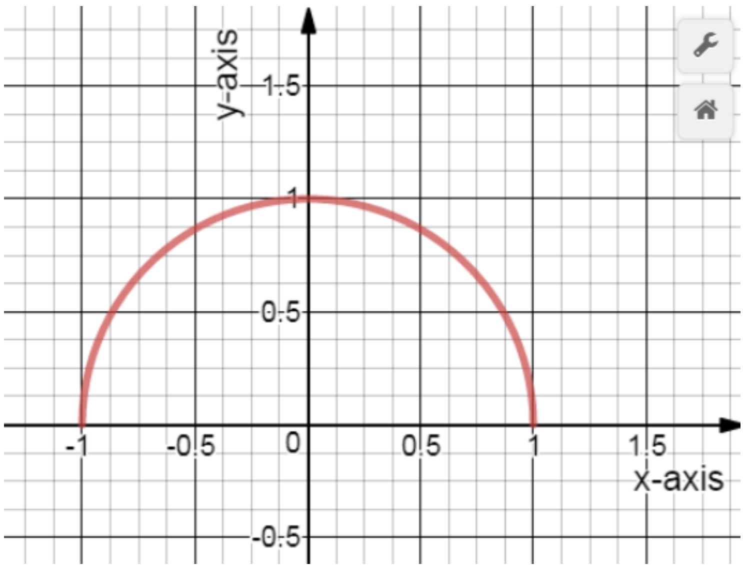


Figure 1.15

Find out if the function f(x) = $\left|\sqrt{1 - x^2}\right|$ is continuous at;

a) The open interval (-1, 1)
b) Closed interval $[-1, 1]$

6. a) Sketch the curve of f(x) = $\sqrt{x^2-1}$.

   b) Describe continuity of f(x) = $\sqrt{x^2-1}$ at 1≤ x.

8. A class is made of students whose performance is represented in the function $f(x) = \frac{x^2-1}{x-1}$ where f(x) is performance and x is number of days the students have been in the class.

   a) Draw the graph of $f(x)$.

   b) Find the value of x 'A missing student' who would make the performance to form a continuous function.

9. Determine the numbers which would make the following functions to be discontinuous.

   a) $f(x) = \frac{x}{x^2+4}$

   b) $f(x) = \frac{x}{4-x^2}$

   c) $f(x) = \left\{\frac{x^2-25}{x-5}\right\}$

Trigonometric limits

To determine limits for trigonometric functions one uses the identity for trigonometry and manipulates them together with algebra and continuity properties.

In previous units we found out that the functions of sine and cosine are all continuous for any real number a.

Activity

i. In groups of three students draw the graph of the following functions at $-360 \leq x \geq 360$ using an interval of 15^0

a. $y = \cos x$

b. $y = \sin x$

c. $y = \tan x$

ii. Examine the graphs, and determine the following limits

a. $\lim_{x \to 0}$ sin x

b. $\lim_{x \to 0}$ cos x

c. $\lim_{x \to 0}$ tan x

Solution

The graphs of these functions are,

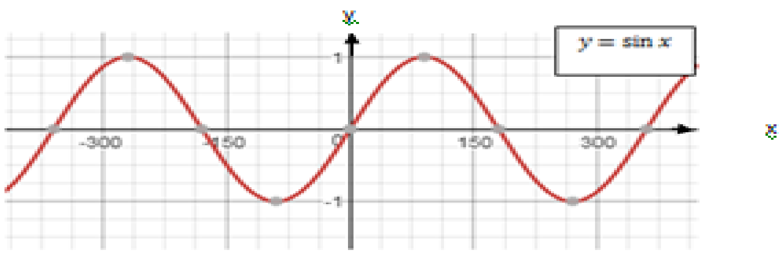


Figure 1.16

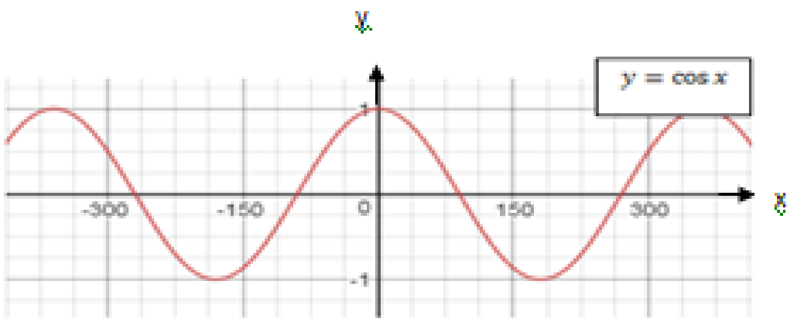


Figure 1.17

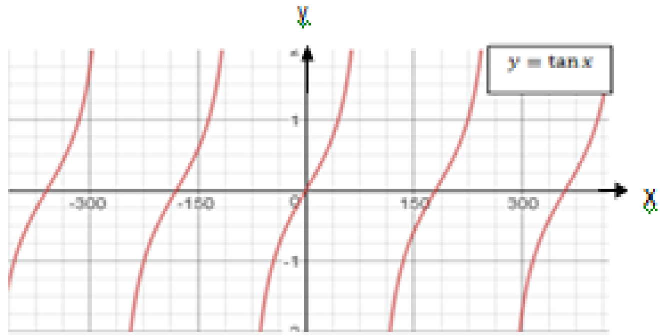


Figure 1.18

Fig 1.16 and 1.17 above illustrates that the graphs of the functions $y = f(x) = cos\, x$ and $y = f(x) = \sin x$ are continuous everywhere.

Figure 1.18 illustrate that $y = f(x) = tan\, x$ is not defined at

$x = (2n + 1)90^0$, $n \in R$ or n = ±1, ±2, ±3, ± 4 , . . .

By observing the graph it is clear that,

a. $\lim\limits_{x \to 0} \sin x = 0$
b. $\lim\limits_{x \to 0} \cos x = 1$
c. $\lim\limits_{x \to 0} \tan x = 0$

The limits of these functions can be obtained by substitution into the function. For any domain a of the function $f(x)$ with a co-domain u . The co-domain $u = f(a)$ is the limit.

In general. $\lim\limits_{x \to a} \cos x = \cos a$

$\lim\limits_{x \to a} \sin x = \sin a$

$\lim\limits_{x \to a} \tan x = \tan a$

Consequently, by substitution

a. $\lim\limits_{x \to 0} \sin x = \sin 0 = 0$

b. $\lim\limits_{x \to 0} \cos x = \cos 0 = 1$

c. $\lim\limits_{x \to 0} \tan x = \tan 0 = 0$

Limits using squeeze theorem

The squeeze theorem states that the graph of two functions g(x) and h(x) with the same limit L at x = c, squeeze a function f(x) between them such that the limit of f(x) at c is L.

Activity

In groups, study figure 1.19 below and answer the questions that follow.

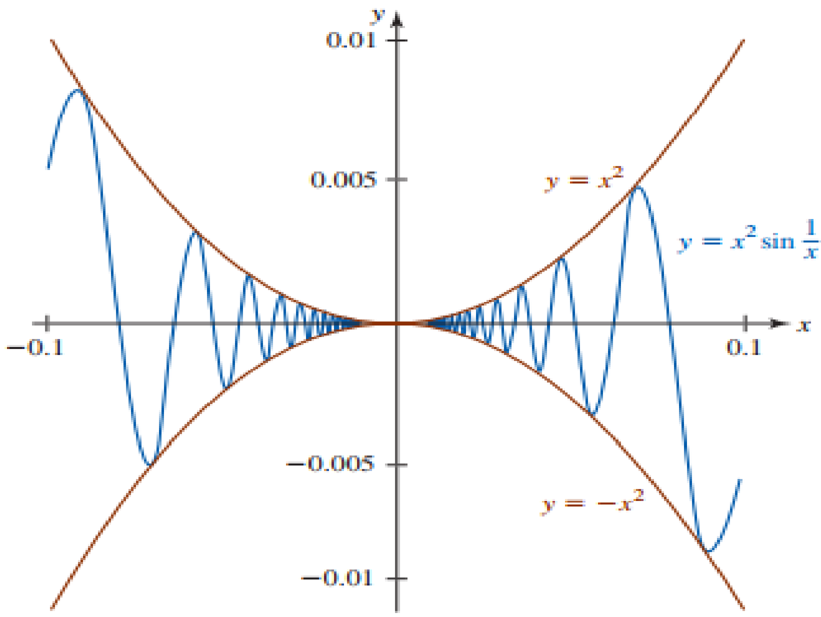


Figure 1.19

a. Determine ,

i. $\lim_{x \to 0} \quad x^2$

ii. $\lim_{x \to 0} \quad -x^2$

b. Of the three functions which function is squeezed by the other functions.

c. Using the graph infer $\lim_{x \to 0} \quad x^2 sin \ {}^{1}/_{x}$

Solution

i. $\lim_{x \to 0} \quad x^2 = 0$

ii. $\lim_{x \to 0} \quad -x^2 \ = \ 0$

$$\lim_{x \to 0} \quad x^2 \ = \lim_{x \to 0} \quad -x^2 = 0$$

In the figure $g(x) \ = \ x^2$ and $h(x) \ = \ -x^2$, have the same limit and squeeze f(x) = x^2 sin ${}^{1}/_{x}$ between them . the squeezed function $\lim_{x \to 0} x^2 sin \ {}^{1}/_{x}$ *hence* have the same limit according to the squeeze theorem. Therefore, $\lim_{x \to 0} x^2 sin \ {}^{1}/_{x} = 0.$

<u>**Note**</u>: the f(x) = $x^2 \, sin \ {}^{1}/_{x}$ is not a composite function and hence

$$\lim_{x \to 0} \quad f(x) \neq \lim_{x \to c} \ x^2 \, . \lim_{x \to c} \ sin \ \ {}^{1}/_{x}$$

Recall, by numerical approach and graph illustrated in figure 1.9 on page 27 it was proven that the

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

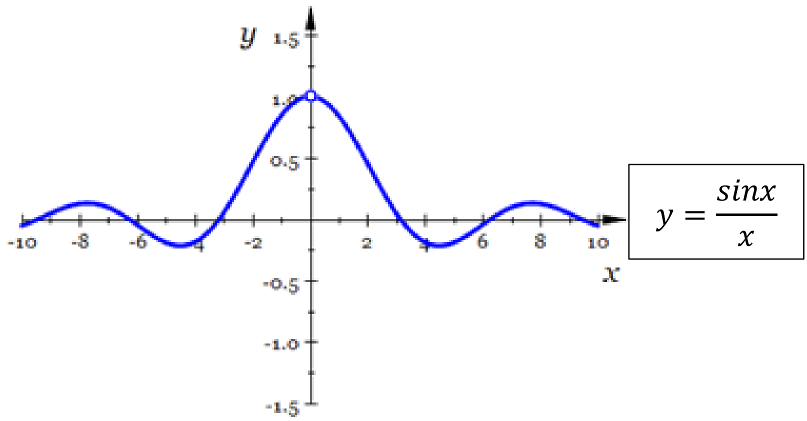


Figure 1.9

Calculating trigonometric limits

The two limits below are used to determine most limits of trigonometric functions as shown below

i) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

ii) $\lim\limits_{x \to c} x^2 sin\left({}^{1}/_{x}\right) = 0$

It's possible to determine limits of several trigonometric functions.

Example 1

a. In pairs discuss and rewrite the expression $\frac{2x^2 - 2x^2 sinx}{x}$ as an expression with a factor of $\frac{\sin x}{x}$ as the only variable.

b. Find the value of

$$\lim_{x \to 0} \frac{2x^2 - 2x^2 sinx}{x}$$

Solution

$a = \frac{7x^2 - 2x\,sinx}{x^2}$ decompose into fractions

$= \frac{7x^2}{x^2}$ - $\frac{2x\,sinx}{x^2}$ simplify

$=$ 7 - $2\,\frac{2sinx}{x}$

b. $\lim_{x \to 0} \frac{7x^2 - 2x^2 sinx}{x}$ $= \lim_{x \to 0}(\ 7 - 2\frac{2sinx}{x})$

$= \lim_{x \to 0}$ 7- 2 $\lim_{x \to 0} \frac{sinx}{x}$

= 7 - 1 = 6

Example 2

In groups, find the $\lim_{x \to 0} \frac{3 \sin 2x}{x}$

Solution

$$\lim_{x \to 0} \frac{3 \sin 2x}{x} = 3\lim_{x \to 0} \frac{\sin 2x}{x}$$

Expanding sin 2x using identify $\sin 2\theta = 2\ \sin\theta \cos\theta$

$\cos 2x = 2 \sin x \cos\theta$ Substituting in a we get

$$3 \lim_{x \to 0} \frac{2(\sin \mathrm{x} \cos\theta)}{x}$$

Rearranging to get $\frac{\sin x}{x}$

$$= \quad 6 \lim_{x \to 0} \cos x . \frac{\sin x}{x}$$

$$= \quad 6\left(\lim_{x\to 0}\cos x \,.\, \lim_{x\to 0}\frac{\sin x}{x}\right)$$

By substituting in cos x for $\lim_{x\to 0}\cos x = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$$= 6 \ (1\times 1)$$

$$= 6$$

Example 1.8

Work in pairs.

Find the value of each of the following

1. $\lim_{x\to 0}\cos x.\frac{10x-3\sin x}{x}$
2. $\lim_{x\to 0}\frac{\sin x}{x}$
3. $\lim_{x\to 0}\frac{\tan x}{x}$
4. $\lim_{x\to 1}\frac{\sin(x-1)}{2x+x^2-3}$
5. $\lim_{x\to 0}\frac{1-\cos x}{x}$
6. $\lim_{x\to 0}\frac{\cos x}{sinx-3}$
7. $\lim_{x\to 0}\cot x$
8. $\lim_{x\to 0}\frac{\sin 4x}{x}$
9. $\lim_{x\to 0}\frac{sec\, x-1}{x}$
10. $\lim_{x\to 0}\frac{\sin 6x}{5x}$
11. $\lim_{x\to 0}\frac{1-\cos x}{x}$
12. $\lim_{x\to 0}\frac{sin7x}{x}$
13. $\lim_{x\to -3}\frac{\sin(x+3)}{x^2+7x+12}$
14. $\lim_{x\to 0}\frac{\sin IxI}{x}$
15. $\lim_{x\to 0}\frac{x}{\tan x}$
16. $\lim_{x\to 0}\frac{x}{cscx}$
17. $\lim_{x\to 0}\frac{sin^3 3x}{x\ sinx^2}$
18. $\lim_{x\to 0}\frac{\sin 3x}{\sin 8x}$
19. $\lim_{x\to 0}\frac{\tan 3x}{x}$

Limits at infinity

The word infinity is used in mathematics to mean a number that is endless, boundless and greater than any assignable quantity or countable number. It is used in concept that something continues forever. Its existence is illustrated graphically by use of asymptotes.

Note that there are both countable (natural numbers, integers, rational numbers) and uncountable infinities.

If a co-domain of a function is an infinity these co-domain is illustrated using asymptotes as learnt earlier.

Example

Draw the graph of f(x) = $\frac{x+1}{2x}$

Solution

x	-3	-2	-1	0	1	2	3
$f(x)$	00.3	0.25	0		1	1.5	2

Undefined

To investigate what happens as $x \to 0, f(x)$ largely increases and tends to infinity as illustrated by figure 1.22 below.

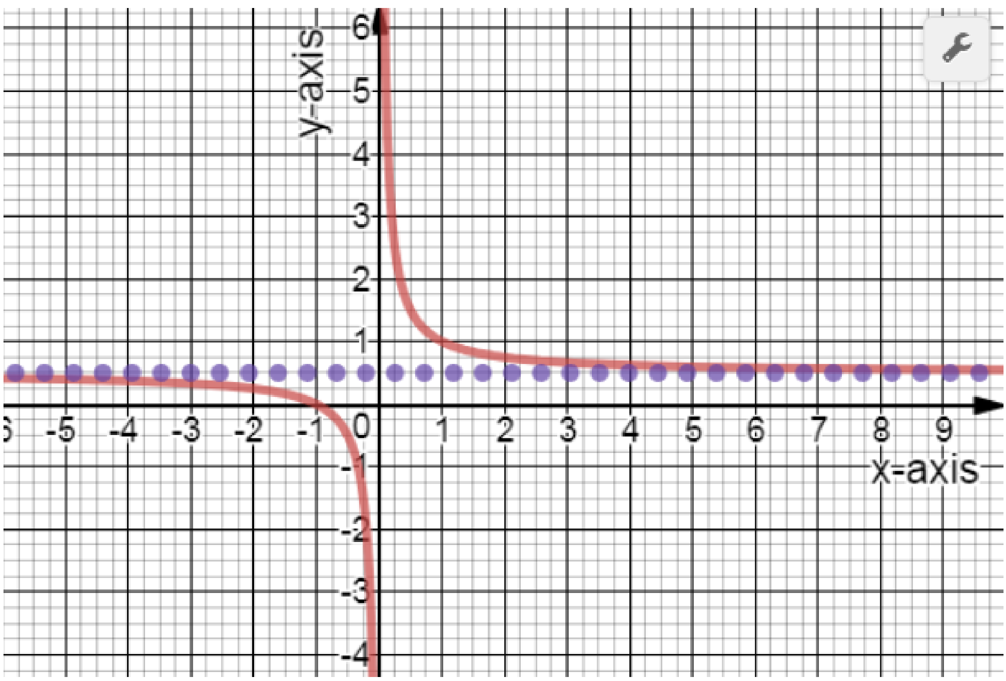


Figure 1.22

You also notice the value of $f(x)$ becomes arbitrary small as x increases and as it decrease without bound the value of $f(x)$ approaches as x increase without bound.

½ is called the limit bound of $f(x)$ as x approaches zero. Hence

$$\lim_{x\to 0}\frac{x+1}{2} = 1/2$$

If a function $f(x)$, L_1 and L_2 are real numbers values then

$\lim\limits_{x\to -\infty} = L_1$ L_1 is limit of f(x) as $x \to -ve\ infinity$

$\lim\limits_{x\to -\infty} = L_2$ L_2 is limit of f(x) as $x \to +ve\ infinity$

Consider,

$$f(x) = \frac{1}{x}$$

As x approaches infinity value of $f(x)$ decreases and approach zero. This is even much faster for

$f(x) = \frac{1}{x^n}$where n is an integer hence

$$\lim_{x\to -\infty}\frac{1}{x^n} = 0 \text{and} \lim_{x\to -\infty}\frac{1}{x^n} = 0$$

The limits at infinity can be determined using the following methods;

a) Graphically
b) Numerically
c) Substitution or by
d) Dividing out

The numerical method has been discussed before, in this area we will discuss the substitution and dividing out method.

Example

1. Find $\lim_{x\to\infty} 14\text{-}\ \frac{1}{x}$.

Solution

By substitution $\frac{1}{\infty} \cong 0$

Hence $\lim_{x\to\infty} (14\text{-}\ \frac{1}{x}) = 14 - 0$

=14.

2. Find,

$$\lim_{x\to\infty} \frac{6x^3 + 12x^2 + 3}{3x^3 + 4}$$

Solution

You note by substitution we get undetermined $\frac{\infty}{\infty}$ form. To avoid $\frac{x}{0}$ form divide by highest denominator power x^3 in numerator and denominator.

Hence,

$$\lim_{x\to\infty} \frac{(6x^3 + 12x^2 + 3) \times \frac{1}{x^3}}{(3x^3 + 4) \times \frac{1}{x^3}}$$

$$\lim_{x\to\infty} \frac{6 + \frac{12}{x} + \frac{3}{x^3}}{3 + \frac{4}{x^3}}$$

By substituting $\frac{k}{\infty} \to 0$ hence

$$\lim_{x\to\infty} f(x) = \frac{6 + 0 + 0}{3 + 0} = \frac{6}{3} = 2$$

Exercise 1.9

Work in groups.

1. Find the limits in each of the following,

   a) $\lim_{x \to \infty} 40 - \frac{3}{x^4}$

   b) $\lim_{x \to \infty} \frac{2x+3}{2x^2+1}$

   c) $\lim_{x \to \infty} \frac{3x^2+3}{x^2+1}$

   d) $\lim_{x \to \infty} \frac{-2x^3+3}{3x^2+7}$

   e) $\lim_{x \to \infty} \frac{6x^2+2x+1}{3x^2+x-2}$

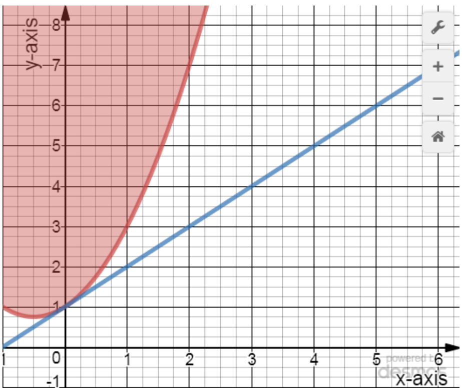
2. The cost of manufacturing success card in a company entail a fixed charge of SSP 5000 and a cost of SSP 0.5 per card.
   a) Write an expression of total cost (T) for making the card. Use x=number of cards.
   b) Write an expression of making an average card for a customer. (T)
   c) Calculate the limit of average cost T as x approaches infinity?

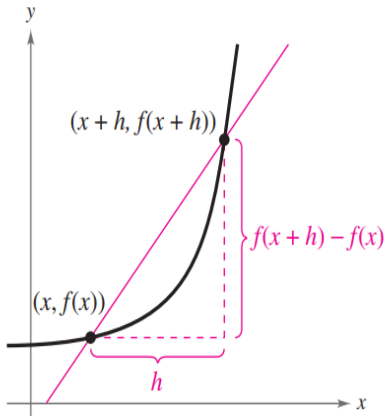
Limit and gradient of curves

The gradient of a curve can be obtained in various ways. Previously geometrical method and calculations using general formula for derivatives were discussed. In commercial method, a tangent is constructed and from Pythagoras theorem used to estimate the value of gradient from constructed right-angled triangle. At microscopic observation such as a tangent may be noted to be a secant.

In this unit we will use limit to derive derivatives. At more systematic method draw a secant line of a curve through a point of tangency$(x, f(x))$.

And another point h units on x-ordinate as shown in the figure 1.23 below.





On magnification of tangent to a curve the two appear as a curve and secant.

Figure 1.23

The secant above passes through $(x, f(x))$and$[(x+h, f(x+h)]$. Using these points we can obtain the gradient of the secant which is an approximation of gradient of the function f(x).

Gradient $= \frac{f(x+h)-f(x)}{(x+h)-x}$

Gradient $= \frac{f(x+h)-f(x)}{(x+h)-x}$

Gradient $= \frac{f(x+h)-f(x)}{h}$

As the value of h reduces and approaches 0 the secant approaches the gradient and it eventually reflect the gradient as shown in graphs in figure 1.24 below.

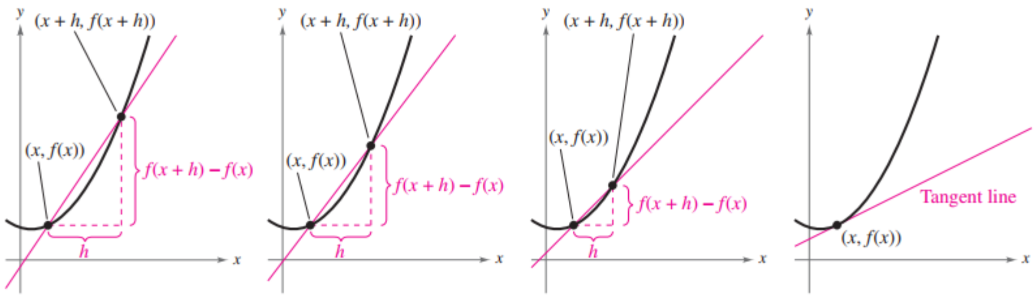


Figure 1.24

The gradients of the secants hence approaches gradient of the function as $h \to 0$ then

Gradient of secant = gradient of tangent = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Using limits the gradient of a function f(x) at a point (x, f(x)) can hence be defined as;

$$\boxed{m = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

This formula is called limit differentiation formula.

Example 1

Find the gradient function of $f(x) = x^2$.

Solution

Using $(x, f(x)) and ((x+h), f(x+h))$

$$\text{Gradient} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$=\lim_{h\to 0} \frac{x^2+2xh+ h^2-x^2}{h}$$

$$=\lim_{h\to 0} \frac{2xh}{h}$$

$$=\lim_{h\to 0} 2x$$

$$= 2x$$

Example 2

In pairs, find the slope of the function $f(x) = x^2+2$ at $(1,3)$

Solution

$$M =\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)+2)^2-(1+2)}{h}$$

$$=\lim_{h\to 0}\frac{(h+3)^2-(3^2+2)}{h}$$

$$= \lim_{h\to 0}\frac{h^2+6h+9-(9+0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2+6h+9-9}{h}$$

$$=\lim_{h\to 0}\frac{h^2+2h}{h}$$

$$=\lim_{x\to h}\frac{h(h+2)}{h}$$

$$= \lim_{h\to 0} \mathrm{h}+2$$

$$= 0+2 = 2$$

Exercise 1.10

Use limits to solve the following questions.

1. Find the slope of the graph of f(x)=x^2 at (-2,4)
2. What is the slope of $f(x) = +4 - 2x$
3. a) Given the formula of a variable y against x is f(x) x^2+1 determining the gradient function of y.
   b) Find the slope of the curve ar x=-1
   c) Find the slope of the curve at y=5
4. Find the derivative of the following functions using limits
   a) $f(x) = x^3$
   b) $f(x) = x^4$
   c) $f(x) = 12x^3 + 12x^3$
   d) $f(x) = x^2 - 4x + 3$
   e) $f(x) = x^2 - 6x + 4$
5. A balloon in a metrological station is projected up. Its height (h) in m above the ground is represented by a function $h(t) = 16t^2 + 64t + 80$ where t is time in hours from the top of the ground, as shown in figure 1.25 below.

Figure 1.25

   a) Find the formula of velocity.
   b) After how long did the balloon stop the movement up? Explain.
   c) What is the maximum height reached by the balloon

Limit and the area under a curve

Consider geometric series as where;

G.S = a, + a, r+ar^2+ar^3+…

As $\sum_{i=1}^{n} a_1 r^{i-1} = \frac{a_1}{1-r}$ *where* $|r| < 1$

GS= $\lim_{n\to\infty} \sum_{i=1}^{n} ar^{i-1}$

G.S = $\lim_{n\to\infty} \frac{a_1(1-R^n)}{1-r}$

$= \frac{a_1}{1-r}$

For $\lim_{x\to\infty}$ r^n=0 for 0< $|r| < 1$

The following properties are also discussed in previous units on summation. Summation and rational forms of series.

1. $\sum_{i=1}^{n} \mathrm{c} = \mathrm{cn}$
2. $\sum_{i=1}^{n} \mathrm{i}^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^{n} \mathrm{i} = \frac{n(n+1)}{2}$
4. $\sum_{i}^{n} \mathrm{i}^3 = \frac{\mathrm{n}^2(\mathrm{n}+1)^2}{4}$
5. $\sum_{i}^{n} (\mathrm{a_{i+}b_i}) = \sum_{\mathrm{i}}^{\mathrm{n}} \mathrm{ai} + \sum_{\mathrm{i}}^{\mathrm{n}} \mathrm{bi}$
6. $\sum_{i}^{n} \mathrm{kai} = \mathrm{k}\sum_{\mathrm{i}}^{\mathrm{n}} \mathrm{ai}$ where a is a constant

In solving problems we need to express them to rational form first and then simplify for n.

Example 1

Find the value of:

$$\sum_{i=1}^{100} \mathrm{i}$$

Solution

$$\sum_{i=1}^{100} \mathrm{i} = \frac{\mathrm{n}(\mathrm{n}+1)}{2}$$

$$= \frac{100(100+1)}{2}$$

$$= 50 \times 101$$

$= \underline{\underline{5050}}$

Example 2

Find the value of

$$\mathrm{S} = \sum_{i=1}^{100} \frac{\mathrm{i}+2}{\mathrm{n}^2} + \frac{3}{\mathrm{n}^2} + \frac{4}{\mathrm{n}^2} + \frac{5}{\mathrm{n}} + \cdots \frac{\mathrm{n}+2}{\mathrm{n}^2}$$

a) Express S as a rational number.
b) Solve for S when n=20

Solution

a) $\mathrm{S} = \sum_{i}^{n} \frac{\mathrm{i}+2}{\mathrm{n}^2}$

$$S = \frac{1}{n^2}\sum_{i}^{n}(i+2)$$

$$S = \frac{1}{n^2}\left(\sum_{i}^{n} i + 2n\right), \qquad \text{but } \sum_{i}^{n} i = \frac{n(n+1)}{2}$$

$$S = \frac{1}{n^2}\left(\frac{n(n+1)}{2} + 2n\right)$$

$$S = \frac{1}{n^2}\left(\frac{n^2+n+4n}{2}\right)$$

$$S = \frac{1}{n^2}\left(\frac{n^2+5n}{2}\right) = \frac{n(n+5)}{2n^2}$$

$$\therefore s = \frac{n+5}{2n}$$

b) At $n = 20$ $s = \frac{20+5}{2\times 20} = 5/8$

The concept of summation of a function is used to determine area under a curve and above x-axis.

Area under a curve

Consider determining area on the figure 1.26 below between f(x) and line x=1 and x=b and x-axis.

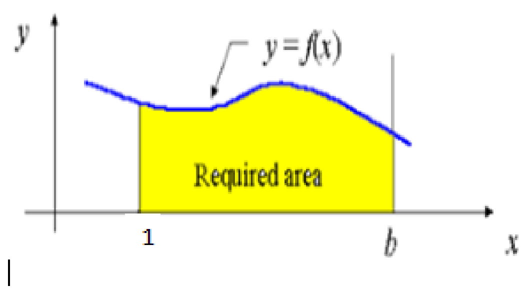


Figure 1.26

If the figure is subdivided into n-rectangles then each rectangle has **width**$\frac{b-1}{n}$

Using height of the rectangles as $f(x)$ for $\quad x = \frac{(b-1)i}{n} + 1$

Where i=1, 2, 3..., n. hence the **height** of each rectangle is f (1+$\frac{(b-1)i}{n}$)

The area of each rectangle is hence

Width x height

$$\frac{(b-1)i}{n} \,.f\left(1 + \frac{(b-1)i}{n}\right)$$

The area of all rectangles is hence

$$\mathrm{A} = \sum_{i=1}^{n} \left(\frac{\mathrm{b}-1}{\mathrm{n}}\right) \mathrm{f}\left(1 + \left(\frac{(\mathrm{b}-1)}{\mathrm{n}}\right)\mathrm{i}\right)$$

The smaller the width the accurate the area. Hence accurate area is limit as n tend to zero.

$$\mathrm{A} = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\left(\frac{\mathrm{b}-1}{\mathrm{n}}\right)}_{\text{Width}} \mathrm{f}\underbrace{\left(1 + \left(\frac{(\mathrm{b}-1)}{\mathrm{n}}\right)\mathrm{i}\right)}_{\text{height}}$$

Example 1

Using five rectangles approximate the area bounded by curve of

$$f(x) = 16 - x^2, \qquad x = 0, \quad x = 2 \text{ and } \quad y = 0.$$

Solution

The sketch of the graph.

Using

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n} . f\left(a + \frac{(b-a)}{n} \right) i \right)$$

$n = 5, b = 2, a = 0, \frac{b-a}{n} = \frac{2-0}{5} = \frac{2}{5}$

Substituting in general formula

$$A = \lim_{n\to\infty} \sum_{i=1}^{5} \frac{2}{5} . f\left(0 + \frac{2}{5} i \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{5} \frac{2}{5} . f\left(\frac{2}{5} i \right)$$

$$= \frac{2}{5} \lim_{n\to\infty} \sum_{i=1}^{5} \frac{2}{5} . f\left(16 - (\frac{2i}{5})^2 \right)$$

$$= \frac{2}{5} \left(\sum_{i}^{5} 16 - \frac{4}{5} . \sum_{i-1}^{5} i^2 \right)$$

$$= \frac{2}{5} \left(80 - \frac{44}{5} \right) = 7\,\frac{12}{25} = 28.48$$

A move accurate is obtained, for instance as $n \rightarrow \infty$ as shown in the example below.

Example 2

Find the area bounded by $f(x) = x^2$, $x = 0$, $x = 1$ and $y = 0$ using limits.

Solution

Consider the figure 1.27 below.

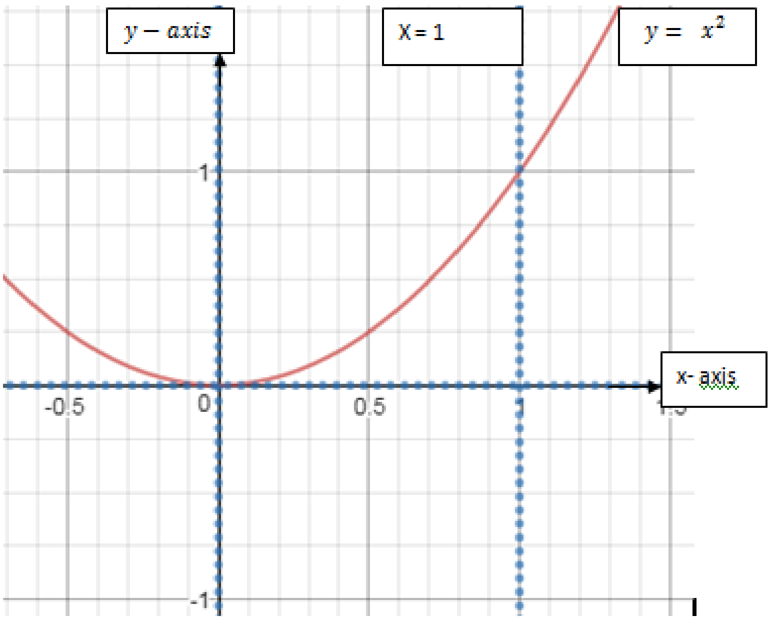


Figure 1.27

Using the **<u>width</u>** $= \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{h}$

<u>Height.</u> $f\left(a + \frac{1}{n} i\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$

$$A = \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)i}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1^2}{\mathrm{n}^2}$$

$$= \sum_{i=1}^{n} \frac{\mathrm{i}}{\mathrm{n}^3} = \frac{1}{\mathrm{n}^3} \sum_{i=1}^{n} \mathrm{i}^2$$

$$\mathrm{A} = \lim_{n\to\infty} = \frac{1}{\mathrm{n}^3} \left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\mathrm{A} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2\mathrm{n}^3 + 3\mathrm{n}^2 + \mathrm{n}}{6\mathrm{n}^3}$$

$$\mathrm{A} = \lim_{n\to\infty} = \frac{2\mathrm{n}^3 + 3\mathrm{n}^2 + \mathrm{n} \times \frac{1}{n^3}}{6\mathrm{n}^3 \times \frac{1}{n^3}}$$

$$\mathrm{A} = \lim_{n\to\infty} = \frac{2 + \frac{3}{\mathrm{n}} + \frac{1}{n^2}}{6}$$

As $\mathrm{n} \to \infty, \frac{3}{n} = 0, \frac{1}{n^2} = 0$

$= \frac{2}{6}$

$\mathrm{A} = \frac{1}{3}$ units.

Exercise 1.11: Work in groups.

1. Evaluate each of the following.

a. $\sum_{1-i}^{n} (i)^4 + 4$

b. $\sum_{1-i}^{n} 8(i)^2$

c. $\sum_{1-i}^{n} 18i$

d. $\sum_{1-i}^{n} (2i + 5)$

2. Evaluate the limits below.

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2\mathrm{i}}{\mathrm{n}^2} + \frac{2}{n}\right)$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{(\mathrm{i})^3}{\mathrm{n}^4} + \frac{5}{n}\right)$

3. The weight (W) in kg of a baby child over several years (t) is represented by equation $\mathrm{w} = 3\mathrm{t} + 3$. A doctor drew the curve of the weight against time for the baby.
   a. Draw the graph of the $\mathrm{w} = 3\mathrm{t} + 3$
   b. Calculate the area bounded by the curve for $t = 0$ to $t = 3$ for the limit $t \to \infty$.
   c. Calculate the height of the baby at 2 years if the height of the baby is given by expression.

$$\mathrm{h\ (in\ meters)} = \frac{Area}{11.25t}$$

4. Using limit, determine the area under curve of $f(x) = x^2 + 5$, in the x interval $[1,3]$ and $y = 0$.

Further reading

Discuss in groups of five students how Desmos and Geogebra software can be used to draw various types of graphs. Make notes and give at least five solved examples of in each of this software.

UNIT 2

TRIGONOMETRY 2

Trigonometric equations and functions

In this unit knowledge of trigonometry learnt in Additional Mathematics Secondary 3 is required to solve trigonometric equations.

Students are required to recall trigonometric identities discussed in previous units and applying them in solving trigonometric equations sum and difference of angles and functions $a\cos\theta + b\sin\theta$ are also discussed.

To solve trigonometric equations we use the graphs of their functions. If the equation has an acute angle expressed or that can be expressed inform of 0º, 30º, 45º, 60º and 90º the equations can be solved without drawing the graphs.

The units include degrees and the radians.

The radian measure

Example 1

In groups,

i. Draw a circle of any radius r units on a graph and mark its center O(0,0)
ii. Mark an arc AB = r units on the circle that subtend an angle θ at the centre of the circle (1 radian)
iii. Define the radian measure.
iv. Calculate the value of angle θ in degree.

Solution

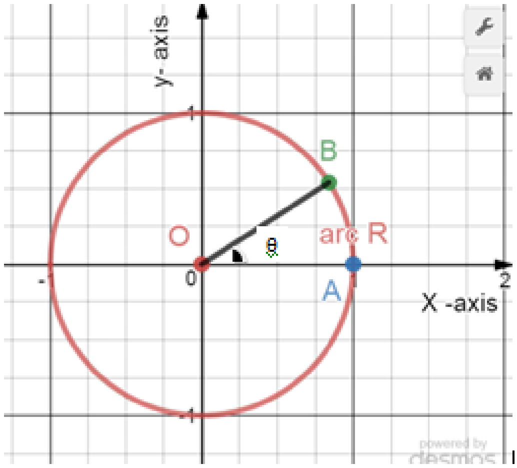


Figure 2.1

iii. Radian measure is an angle θ subtended at the centre of a circle radius R by an arc R-units. The symbol for radian is a power of c (1^c)

iv. To obtain θ

The circumference subtends an angle of 360^0 at the center and hence,

$2\Pi r = 360^0,$

$$r = \frac{360^0}{2 \times 3.14} = 57.2727^0 \ (4d.p)$$

The radius r subtends an angle 57.2727^0 at the center of the circle. Similarly

$2\Pi r = 2\Pi radian = 2\Pi^c = 360^0$, simplifying this we get

In general,

$1\ Radian = 1^c = 57.2727^0$

$\Pi^c = 180^0,$

Example 2

In pairs:

i. Express 120^0 in radian
ii. Express 240^0 in form of Π^c
iii. Express $\frac{\Pi^c}{4}$ into degree

Solution

i. $57.2727^0 = 1^c$

$$120^0 \rightarrow = \frac{120^0}{57.2727^0} \times 1^c = 2.0952^c$$

ii. $180^0 = \Pi^c$,

$$240^0 \rightarrow = \frac{240^0 \times \Pi^c}{180^0} = \frac{4\Pi^c}{3}$$

iii. $\Pi^c = 180^0$,

$$\frac{\Pi^c}{4} \rightarrow = \frac{180^0 \times \frac{\Pi^c}{4}}{\Pi^c} = 45^0$$

Example 3

i. In groups, draw an isosceles right angle triangle and an equilateral triangle.
ii. Label all the angles in the triangle.
iii. Use the triangle to complete the table below.

Angle in degree (θ)	0	30	45	60	90
Angle in $\Pi^c(\theta)$					
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					

The ratio for these angles was discussed in additional mathematics secondary 3. Table 2.1 below shows the trigonometric ratios of some special angles

Table 2.1

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

For non-acute angles have equivalent acute angles to angles outside $[0°, 90°]$that they may be obtained by making respective revolutions.

Simple trigonometric equations solutions

Trigonometric equation is an equations whose variables are unknown angles expressed in trigonometric ratios. For instance $\sin\theta^0 = ½$

The solutions of trigonometric equations are obtained by use of trigonometric equation graphs or computation similar to the algebraic simplification if the angle can be expressed as a special angle.

Example 4

a. On the same axis draw the graph of,
   i. $y = \sin x$
   ii. $y = 0.5$
b. Using the graph solve $\sin x = 0.5$ for $0^0 \leq x \leq 360^0$

Solution

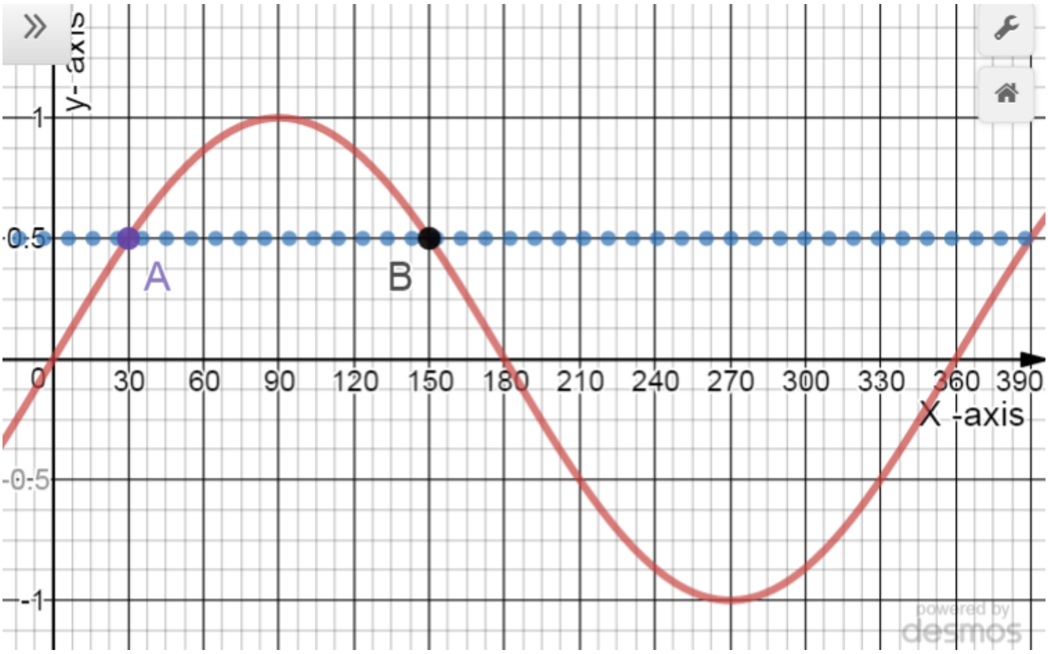


Figure 2.2

The graph of function of $y = sin\, x$ $\ and\ y = 0.5$ form simultaneously equation at point of intersection in which $0.5 = sin\, x$.

$y \ = \sin x \ldots \ldots \ldots \ldots \ldots 1.$

$0.5 \ = \sin x \ldots \ldots \ldots \ldots 2.$

$y - 0.5 = 0$

$y = 0.5$ This implies that, at point A and B, x- ordinate is the solution for x in $\sin x = 0.5$

From the graph it is hence clear that $x = 30^0$ and 150^0

Example 5

a. In pairs, draw the graph of the following functions on the same axis,

   i. $y = \cos x$

   ii. $y = 0.5$

b. Using the graph solve $\cos x = 0.5$ for $-180^0 \leq x \leq 360^0$

Solution

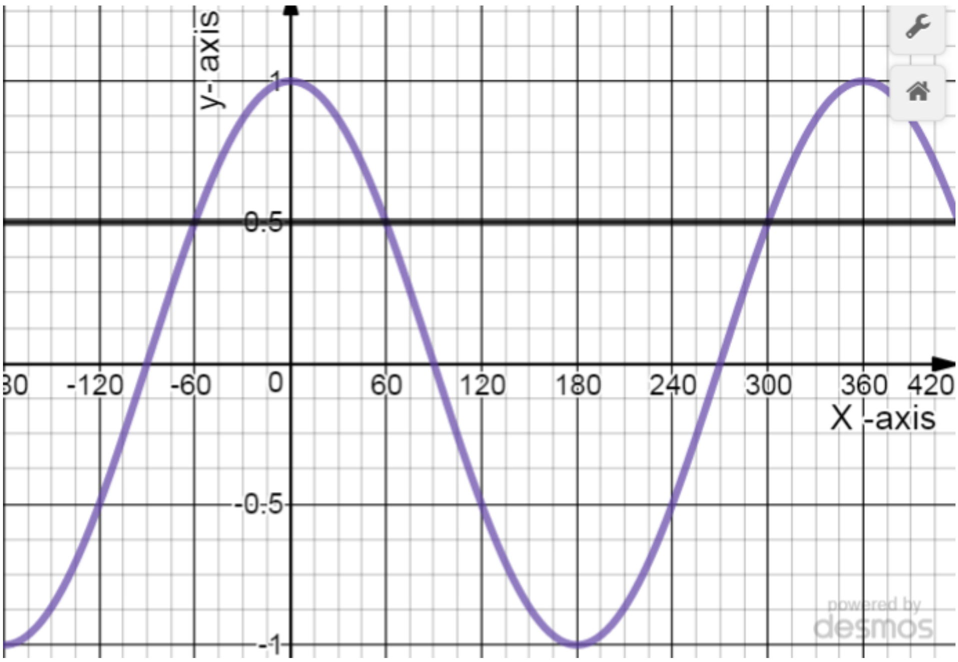


Figure 2.3

Choosing the intersections of the graphs as the solutions of x we get

$x = -60^0, 120^0$ and 240^0

Example 6

Solve for x in $\cos x + 2\,sinx = 1.5$ in the interval $0^0 \leq x \leq 720^0$

Solution

Using the graph of the function $y = \cos x + 2\, sinx$

The function form simultaneous equation with function obtained by subtracting the two equations as shown below.

$$\begin{array}{r} \cos x + 2\, sinx = 1.5 \\ \underline{\quad y = \cos x + 2\, sinx} \\ y = 1.5 \end{array}$$

We then draw the graphs of $y = \cos x + 2\, sinx$ *and* $y = 1.5$ to obtain the simultaneous equationsolutions.

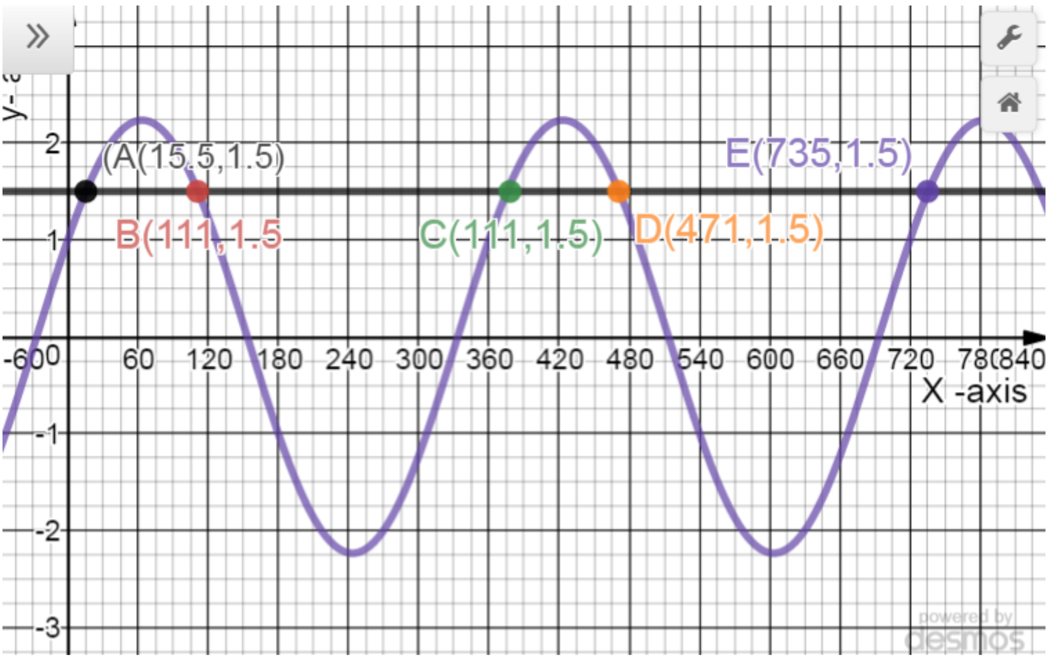


Figure 2.4

Figure 2.4 above shows the graph of $y = \cos x + 2\, sinx$ *and* $y = 1.5$. From the graph $x = 15.6^{50}, 111^0, 375^0, 471^0$ *and* 735^0

Example 7

a. In groups of three students draw the graph of $y = sin\,\mu \quad where\ \mu = 2x$ for $0 \leq x \leq 360^0$ and interval of 15^0

b. Using the graph , solve for x in $\quad sin\,2x = \frac{1}{2}$

Solution

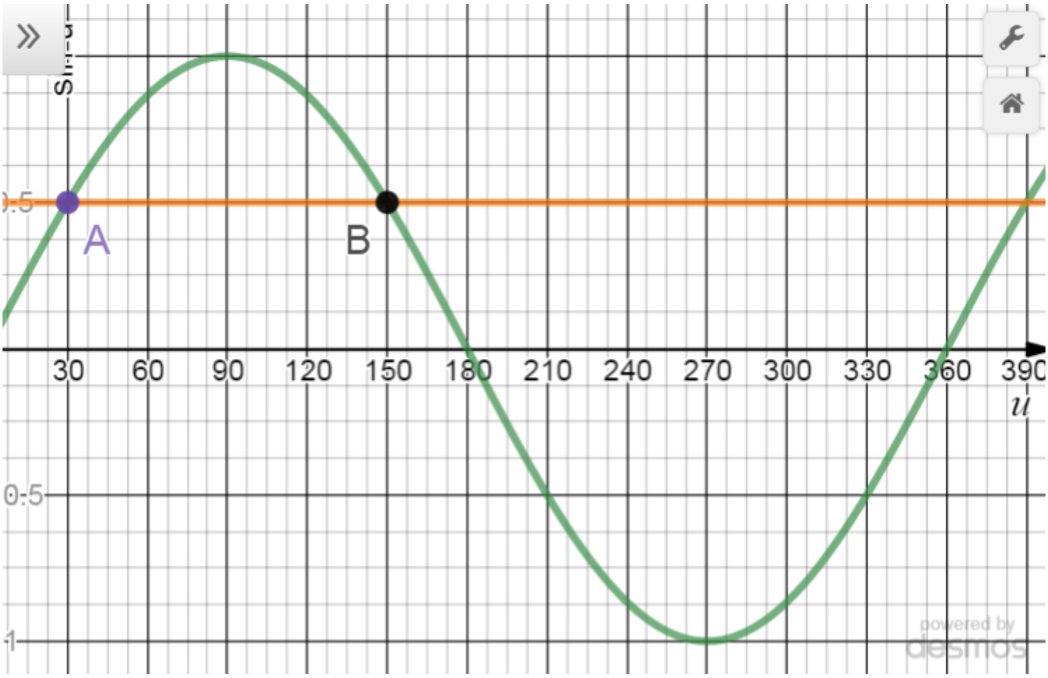


Figure 2.5

From graph it is clear that;

$\mu\ = 30^0,\ 150^0\ and\ 390^0$.

Since $\quad \mu\ = 2x$

$2x = \ 30^0,\ 150^0\ and\ 390^0$ Dividing by 2.

$x = 15^0, 75^0\ and\ 195^0$.

Exercise 2.1

Work in groups.

1. Find the solution to the trigonometric equations below by using graphs of respective equations.
   a) $Tan\ x = -1 \quad -\pi \le x \le \pi$
   b) $Cos2x = \frac{\sqrt{3}}{2} \qquad 0 \le x \le 2\pi$
   c) $Cos\ 2x = \frac{1}{\sqrt{2}} \qquad -180^0 \le x \le 180^0$
   d) $Tan\ 2x = 1 \qquad -90^0 \le x \le 90^0$
   e) $Sin\ 2x = \frac{1}{2} \qquad -180^0 \le x \le 0$
   f) $Cos\left(\frac{1}{2x}\right) = -5\frac{3}{2} \qquad -180^0 < x < 180^0$

2. Find the solution of the following trigonometric equations using Desmos graphs.
   a) $Sin\ (x)\ + 2 = 3$ for $0^0 < x < 360^0$
   b) $7\ tan\ \theta = 2/3 + tan\theta$ for $0° < x < 360^0$
   c) $3(\sin x + 2) = 3 - \sin x \qquad 0° < x < 360^0.$
   d) $\frac{1}{2}(\sec\theta + 3) = \sec\theta + \frac{5}{2}. \qquad 0° < x < 360^0$
   e) $Sin\ (\frac{x}{2}) = \frac{\sqrt{3}}{2} \qquad \frac{-\pi^c}{2} \le x \le \pi^c$
   f) $\tan x = \sqrt{3}. \qquad -180^0 \le x \le 180^0$
   g) $\tan x = -\sqrt{3} \qquad -180^0 \le x \le 180^0$
   h) $Cos\ x = \frac{1}{2} \qquad -180^0 \le x \le 180^0$
   i) $Sin\ x = -\frac{1}{2}. \qquad -180^0 \le x \le 180^0$

3. Using any suitable method solve the following trigonometric equations.
   a) $\tan^3\theta + 3 = 0 \qquad 0° \le x \le 360^0$
   b) $\tan x = \sqrt{3} \qquad 30^0 < x < 2\pi^c$
   c) $\sin 2x = -1 \qquad -\pi^c < x < \pi^c$
   d) $\cos 3x = \frac{1}{\sqrt{2}} \qquad -\pi^c \le x \le \pi^c$

e) $\tan\frac{1}{2}x = -1$ $\qquad -2\pi^c < x < 0$

f) $\sin x = \frac{1}{\sqrt{2}}$ $\qquad 0° \le x \le 360^0$

g) $\cos x = \frac{-1}{2}$ $\qquad 0° \le x \le 360^0$

h) $\tan x = \frac{1}{\sqrt{3}}$ $\qquad 0° \le x \le 360^0$

i) $\cos x = -1$ $\qquad 0° \le x \le 360^0$

4. The voltage v in volts in an electrical circuit given by the formula $v = 20\cos(\pi^c)$ where t is time in seconds.
   a) Draw a graph of voltage v against time for $0 \le t \le 2$ intervals of $\frac{\pi}{12}$
   b) What is the voltage of the electric circuit when 1 sec?
   c) How many ties does the voltage v equal 12 voltage in the first two seconds?
   d) At what time is the voltage 12v?

Solving a quadratic trigonometric equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$ where a, b and c are constant and x is variable.

A quadratic trigonometric equation is an equation of the form

$ax^2 + bx + c = 0$ where the variable x is a trigonometric ratio for example,

$3tan^2\theta + -5\tan\theta - 2 = 0.$

Quadratic trigonometry equation is solved just like the other equations one can use either factorization, completing square, quadratic formula or graphing.

The equation to trigonometric function $ax^2 + bx + c = 0$ and simultaneously identify solution at $y = 0.$

Example 8

In pairs, solve $2\sin^2\theta + \sin\theta - 1 = 0$ for $0° \le x \le 360°$

Solution

Let

$x = sin.$

$x^2 = \sin^2\theta.$

$2x^2 + x - 1 = 0.$

By quadratic formula

$x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}.$

$x = -1 \; or \; x = \frac{2}{4} = \frac{1}{2}.$

Using $\boldsymbol{x = -1}$

$\sin\theta = -1$

$\theta = \sin^{-1} 1.$

$\theta = 0^0.$

The acute angle $\theta = 0^0$ has angles outside the interval $[0, 90^0]$ in the interval $[0, 360^0]$ such that,

$\theta = 0^0, \; 180^0 \; \; and \; 360^0$

Using $\boldsymbol{x = \frac{1}{2}}$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}(\frac{1}{2}).$

$\theta = 30^0$.

The acute angle $\theta = 30^0$ has angles outside the interval $[0, 90^0]$ in the interval $[0, 360^0]$ such that,

$$\theta = 30^0, \qquad and\ 150^0$$

Combining the two set of values we get

$$x = 30^0, 150^0, 180^0\ and\ 360^0.$$

Alternatively one can use the other methods such as graphing, factorization and completing square.

Exercise 2.2

Work in pairs.

Solve the following quadratic trigonometric equations.

1. $2\cos^2 x - \sqrt{3}\cos x = 0 \qquad 0^0 \le x \le 360^0$
2. $\tan^2 x + 3 = 0 \qquad 0^0 \le \theta \le 360^0$
3. $\sec\theta \csc\theta + \sqrt{2}\csc\theta \qquad 0^0 \le \theta \le 360^0$
4. $2\sin x\cos x + \sin x = 0 \qquad 0^0 \le \theta \le 360^0$
5. $4\sin^2 x - 1 = 0 \qquad 0^0 \le x \le 360^0$
6. $2\sin x - 1 = \frac{3}{\sin x} \qquad 0 \le x \le 2\pi$
7. $3\tan^2 x - 5\tan x - 2 = 0 \qquad 0 \le x \le 360^0$

Using identities in solving quadratic equations with more than one trigonometric functions

Quadratic trigonometric equations are trigonometric equations that contain squares in their expressions. For instance, $\cos^2 x +$ $\cos x = \sin^2 x$.

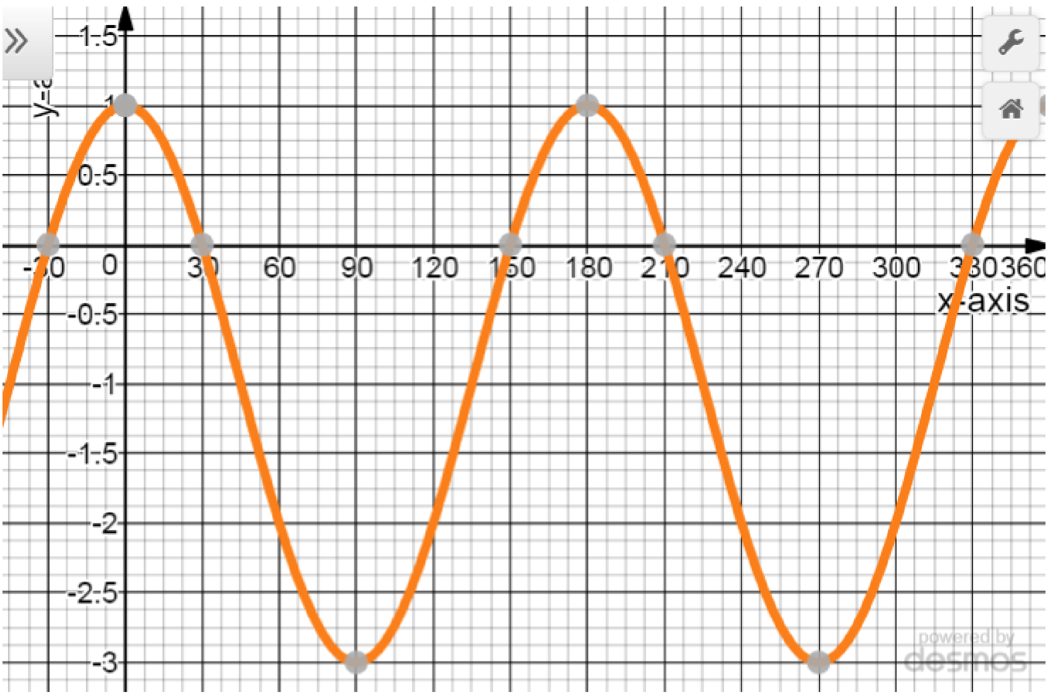
Graphical solutions of Quadratic Trigonometric Equations

Example 9

a. In groups of four students of draw the graph of the function
$y = 3\cos^2 x - \sin^2 x - 2$

b. Use your graph to solve
$3\cos^2 x - 3 = \sin^2 x - 1$ for $-30° \leq x \leq 360^0$

Solution



Using the graph of the function $y = \cos x + 2\, sinx$

The function form simultaneous equation with function obtained by subtracting the two equations as shown below.

$$\begin{array}{l} y \;=\; 3\cos^2 x - \sin^2 x - 2 \\ \underline{\;3\cos^2 x - 3 = \sin^2 x - 1\;} \\ y = 0 \qquad (the\ x - axis) \end{array}$$

The intersection of the graph of $y = 3\cos^2 x - \sin^2 x - 2$
And the x-axis is hence the solution for the equation.

Therefore, $x = -30^0\ 30^0, 150^0\ , 210^0 and\ 330^0$

Example 10

In groups, use graphs to solve for x in,

$\cos 2x + 2\cos^2 x = 2$ for interval $0° \leq x \leq 360^0$

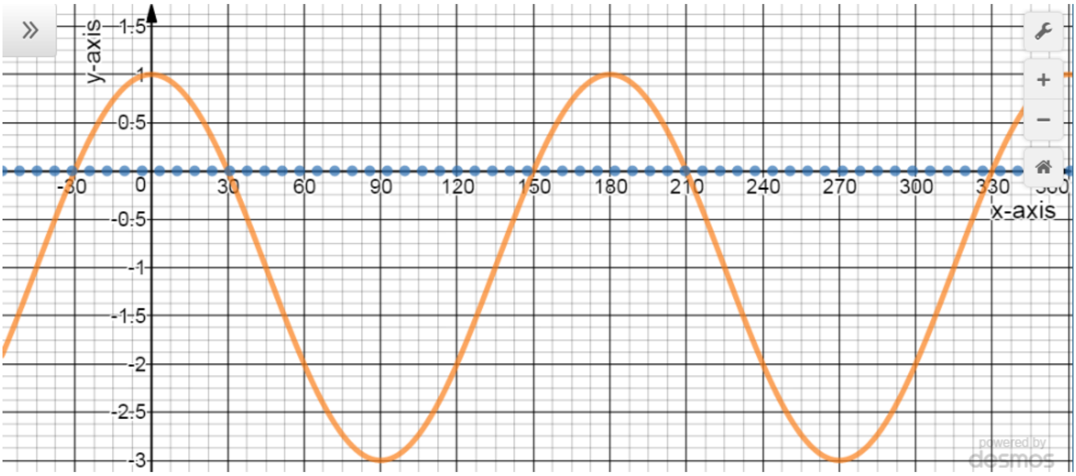
Solution

Re write, $\cos 2x + 2\cos^2 x = 2$ as a quadratic equation

$\cos 2x + 2\cos^2 x - 2 = 0$, express the quadratic equation as a function of y,

$y = \cos 2x + 2\cos^2 x - 2$ Draw the graph of this equation and $y = 0$

The two give simultaneous solutions of $\cos 2x + 2\cos^2 x = 2$



Therefore, $x = 30^0, 150^0\ , 210^0 \, and\ 330^0$

Arithmetic solutions of Quadratic trigonometric equations

If a trigonometric equation has more than one functions trigonometric identities discussed and derived additional mathematics secondary 3 are often used

The tables below shows various identities used.

Table 2.2

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cos^2\theta + \sin^2\theta = 1$
$\csc\theta = \frac{1}{\sin\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$1 + \tan^2\theta = \sec^2\theta$
$\cot\theta = \frac{1}{\tan\theta}$		$\cot^2\theta + 1 = \csc^2\theta$

Table 2.3

Cofunction Identities	Double-Angle Identities	Half-Angle Identities
$\cos\theta = \sin(90^\circ - \theta)$	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}$
$\sin\theta = \cos(90^\circ - \theta)$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$	$\cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}$
$\tan\theta = \cot(90^\circ - \theta)$	$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$	$\tan\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$
$\cot\theta = \tan(90^\circ - \theta)$		
$\sec\theta = \csc(90^\circ - \theta)$		
$\csc\theta = \sec(90^\circ - \theta)$		

Table 2.4

Sum and difference of angle identities
$\sin(\theta + \beta) = \sin\theta\cos\beta \pm \cos\theta\sin\beta$
$\cos(\theta + \beta) = \cos\theta + \cos\beta \pm \sin\theta\sin\beta$

The example below illustrates how identities are used to solve trigonometric equations.

Example 1

Solve $3\cos^2 x - 3 = \sin^2 x - 1$ for $0^\circ \leq x \leq 360^0$

Solution

To have the same trigonometric ratio in the equation use, Pythagoras identity:

$$\sin^2\theta + cos^2\theta = 1.$$

$$sin\,^2\theta = 1 - cos^2\text{Hence}$$

$$sin\,^2x = 1 - \cos^2 x.$$

Substituting in we get,

$$3\,cos\,^2x - 3 = \ sin\,^2x - 1.$$

$$3\,cos\,^22x - 3 = 1 - \cos^2 x - 1.$$

$$3\ cos\ ^2x - 3 = -\cos^2 x.$$

$$3\ cos\ ^2x - 3 + \cos^2 x = 0.$$

$$4cos\ ^2x - 3 = 0.$$

$$cos\ ^2x = \frac{3}{4}.$$

Taking square on both sides the angle is the positive.

$$\cos x = \sqrt{\frac{3}{4}}.$$

$$\cos x = \frac{\sqrt{3}}{2}.$$

$$x = \cos^{-1} \quad \frac{\sqrt{3}}{2}.$$

$x = 30^0$ This the acute angle

$x =$ 150^0 , 210^0 *and* 330^0 for the interval outside $[0, 90^0]$

Therefore $x = 30^0, 150^0 , 210^0 and\ 330^0$

Example 2

Solve the equation $\cos 2\theta + 2\cos^2\theta = 2$for interval $0° \leq \theta \leq 360^0$

Solution

Use double angle formula to remove (2θ)in the trigonometric.

Consider identity

$$cos\ 2\theta = cos^2\theta - sin^2\theta.$$

Substituting back

$$\cos^2\theta - sin^2\theta + 2\ cos^2\theta = 2.$$

$$3\ cos^2\theta - sin^2\theta = 2.$$

To have some trigonometric ratio we need to use Pythagoras identity

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\sin^2\theta = 1 - \cos^2\theta.$$

Substituting basic we get

$$3\cos^2\theta - (1 - \cos^2\theta) = 2.$$

$$3\cos^2\theta - 1 + \cos^2\theta = 2.$$

$$4\cos^2\theta = 3.$$

$$\cos^2\theta = \frac{3}{4}.$$

Taking square root on both sides

$$\cos\theta = \sqrt{\frac{3}{4}}.$$

$$\cos\theta = \frac{\sqrt{3}}{2}.$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2}.$$

$\theta = 30^0$ is the acute angle , outside interval $[0, 90^0]$ we get , 150^0 , 210^0 *and* 330^0

Hence, $x = 30^0, 150^0, 210^0$ *and* 330^0

Alternatively on can draw the graph of the trigonometric function

$y = \cos 2\theta + 2\cos^2\theta - 2$and seek the solution for intersection with line $y = 0$ which are solved simultaneously

Exercise 2.3

Work in groups.

Solve the following trigonometric equations using any suitable method.

1. $5\cos^2\theta - 3 = \sin^2\theta$ for $-180° \leq \theta \leq 180$
2. $\tan^2\theta = 2\sin^2\theta - 3$ for $180° \leq \theta \leq 180$
3. $\cos^2\theta + 3\cos\theta = \sec^2\theta - 2$ for $0° < x < 360^0$
4. $\cos\theta = \frac{\sqrt{3}}{2}$ for $0° \leq \theta \leq 2\pi$
5. $\text{Cos } 2\theta = \frac{\sqrt{3}}{2} 0° \leq \theta \leq 2\pi$
6. $2\sin x + 1 = \csc x$ $0° < x < 360^0$
7. $\cos 2\theta - 2\cos\theta = 0$ $0° < \theta < 360^0$
8. $\sin(90 - x) + 2\cos x = 2$ $0° < x < 360^0$
9. $\cos^2 x + \cos x = \sin^2 x$ $0° < x < 360^0$
10. $\sin x = \sin 2x$ $0° < x < 360^0$
11. $\sin x + \cos x = 1$ $0° < x < 360^0$
12. Deng swam 90metres from a point A to a point B. He then turned by 90º and moved to a point C on East of A as shown in the figure 2.6 below

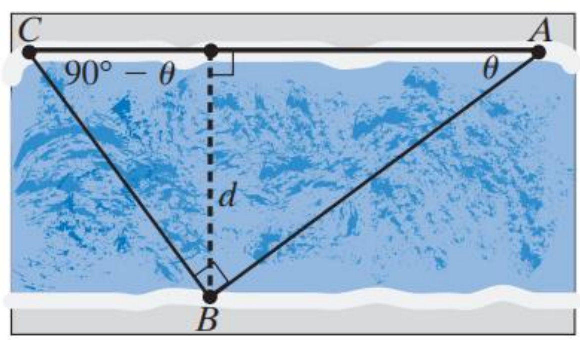


Figure 2.6

Given that d is the shortest distance from B to line AC angle $CAB = \theta^0$ and line $BC = 60$m

a) Express d in terms of
   i. θ
   ii. $90^0 - \theta$

b) Calculate the value of:-
   i. θ
   ii. d

13. An electric pole is supported by two equal wires AB and CD such that $AB = CD = x$. Wire AB make an angle θ^0 to the ground floor while CD make $2\theta^0$ with the ground given the pole height FB = y cm and , $FD = 1.75FB$ as shown in the figure 2.7 below.

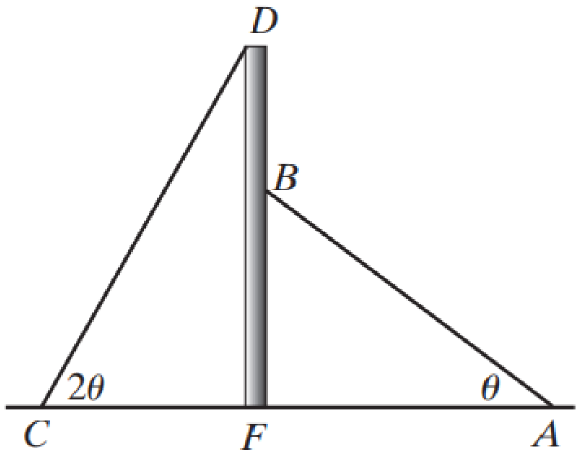
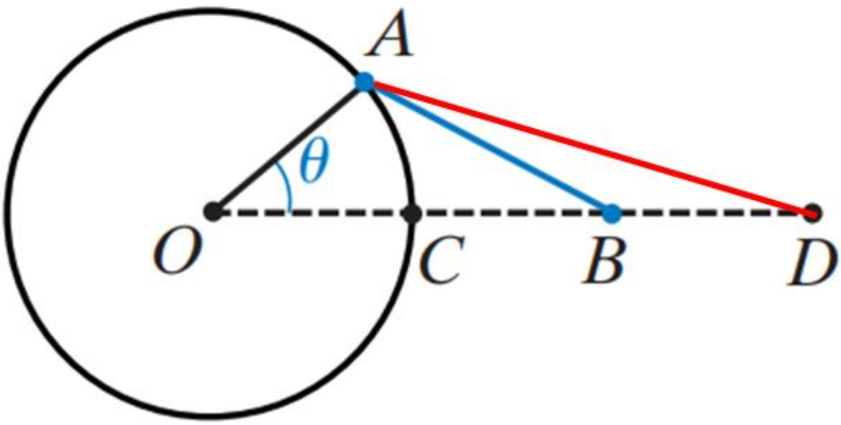


Figure 2.7

a. Express in term of x and y
   i. Sin θ
   ii. Sin2θ
b. Determine angle θ

14. The figure 2.8 below shows a rolling machine made by an engineer such that as the circular disc center O and radius OA

= r m rotate point B slides on CD and line AO make an angle θ with line OD. The movement is restricted such that $<AOD = \theta$ is at interval of $-45^0 \leq \theta \leq 45^0$



Given that radius $OA = 2$m, the length $l = AB = 2$m and the moment of point B is described by the equations

$$CB = r(\cos\theta - 1) + \sqrt{l^2 - r^2\sin^2\theta}$$

Find the value of,

a) CB when:

I. $\theta = 30^0$

II. $\theta = 45^0$

b) θ when:

I. $CB = 2$m

II. $CB = 1.5$m

UNIT 3

CALCULUS 3

Introduction

In the previous units in calculus we have discussed, how to differentiate and integrate some functions definition and determinations of differential functions through limits. In this unit more techniques of differentiation and integrations will be discussed.

In this unit we will discuss use of product rule and quotient rule in differentiation, the implicit differentiation, and integration of trigonometric, logarithmic, exponential functions and functions that contain power of linear functions. Application of integration will also be discussed in details.

Differentiation by the product and quotient rule

The product rule

The production rule is used to differentiate functions that are made of products of two functions. It states that if a function f is made of product of two functions u and v, then the derivative y is $u'v + uv'$ hence

if $y = uv, \quad y' = u'\,v + uv'$

Alternatively if $\quad y = uv, \quad \frac{dy}{dx} = \frac{du}{dx}\ v + u\ \frac{dv}{dx}$

The quotient rule

A quotient is the solution of division. Quotient rule is used to determine the derivative of a function made of a numerator and a denominator. It states

If, $\ y = \frac{u}{v} \qquad y' = \frac{(u'v - uv')}{v^2}$

Alternatively

If $y = \frac{u}{v}$ $\quad \frac{dy}{dx} = (\frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2})$

Activity 1

In groups, find out how the product and quotient rules are derived.

Example 1

In pairs, find the derivative of $y = (3x^2 + 4)(7x^2 + 10x)$.

Solution

$\frac{dy}{dx}(u.v) = \frac{du}{dx}v + u\frac{dv}{dx}$

Let $u = (3x^2 + 4)$, $\frac{du}{dx} = 6x$

$v = (7x^2 + 10x)$, $\quad \frac{dv}{dx} = 14x + 10$ then ,

$y' = 6x(7x^2 + 10x) + (3x^2 + 4)(14x + 10)$

$y' = 42x^3 + 60x^2 + 42x^3 + 30x^2 + 56x + 40$

$y' = 83x^3 + 90x^2 + 56x + 40$

Example 2

In pairs, find the derivate of $y = \frac{x^3 + 2x}{4x+1}$

Solution

Use quotient rule

$$\frac{d}{dx}(\frac{u}{v}) = (\frac{\frac{dy}{dx}v - u\frac{dv}{dx}}{v^2})$$

Then; $u = x^3 + 2x$

$$\frac{du}{dx} = 3x^2 + 2$$

$$v^2 = (4x + 1)^2 = 16x^2 + 8x + 1$$

$$y' = \frac{(3x^3 + 2)\,(4x+1) - (x^3 + 2x)\,(4)}{16x^2 + 8x + 1}.$$

$$y' = \frac{8x^3 + 3x^2 + 2}{16x^2 + 8x + 1}.$$

Exercise 3.1

Work in pairs.

1. Find the derivate of each of the following.
   a) $y = \frac{2x+1}{3x-4}$
   b) $y = \frac{3x-4}{2x+1}$
   c) $y = \frac{x^2-2}{2x+1}$
   d) $y = \frac{2x+1}{x^2-3}$
   e) A graph has a function $y = \frac{x^2+x-2}{x^3+6}$ find the gradient of the curve at $x = 0$
2. Find $\frac{dy}{dx}$ given that $y = \sqrt{x}\,(9 + bx)$
3. Determine the gradient f1(0) of a curve.

   $f(x) = \frac{x^2+6}{2x-7}$
4. Differentiate the following functions.
   a) $y = x^3(4-x)^{1/2}$
   b) $y = 2x^6(1+x)^5$
   c) $y = x^{-2}(1+x)^{1/2}$
   d) $y = \frac{x}{x+1}$

Implicit differentiation

This is a process that involve differentiating implicit functions. A function where y is expressed in terms of x is called an explicit function. If f is an explicit function of x then $y = f(x)$. Implicit function is a function that cannot be rearranged such that y is expressed in terms of x. If an implicit function (g) contain variables x and y it is expressed as $(x, y) = 0$. Some explicit function cannot be expressed as $g(x, y) = 0$ by using basic method of rearrangement. If explicit function can be expressed as $g(x, y) = 0$, it is differentiated as other function. In this unit we will discuss differentiating implicit function of form $g(x, y) = 0$, in this differentiation $\frac{dy}{dx}(y^n) =$ $ny^{(n-1)}\frac{dy}{dx}$

Example 1

In groups, differentiate $x^3 + y^3 - 4 = 0$

Solution

Consider $g(x, y) = 0$ hence

$$\frac{d}{dx}(x^3) + \frac{d}{dx}y^3 - \frac{d}{dx}4 = 0.$$

$$\frac{d}{dx}(x^3) = 3x^2.$$

$$\frac{d}{dx}y^3 = \frac{dy}{dy}(y^3).\frac{dy}{dx} = 3y^2\frac{dy}{dx}.$$

$$\frac{d}{dx}(4) = 0.$$

Hence we get.

$$\frac{d}{dx}g(x, y) = 3x^2 + 3y^2\frac{dy}{dx} = 0.$$

$$3x^2 + 3y^2\frac{dy}{dx} = 0.$$

$$3y^2\frac{dy}{dx} = -3x^2.$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}.$$

Exercise 3.2

Differentiate the following implicit functions with respect to x for questions 1→10.

1. $(x-y)^2 = x + y + 1$
2. $x^2y^3 + x^3y^2 = 0$
3. $x = 3 + \sqrt{x^2 + y^2}$
4. $\frac{x-y^3}{y+x^2} - (x+2) = 0$
5. $4x^2 = 2y^3 + 4y$
6. $\frac{y}{x^3} + \frac{x}{y^3} - x^3y^4 = 0$
7. $5y^2 = 2x^3 - 5y$
8. $x^2 + y^2 = \sqrt[3]{8x^2 + y^2}$
9. $2x^3 = 2y^2 + 5$
10. $3x^2 + 3y^2 = 2$
11. Determine the equation of the tangent to line $x^2 - 9 = (y-x)^2$ at a point (1, 9)
12. The movement of a car from a point A is represented by implicit function $5 = 4t^2 + 5h^2$ where h is distance (metres) and t is time. Find the:
    a) Expression for acceleration.
    b) Acceleration at $t = 1$

Derivative of exponential and logarithmic functions

A logarithmic function is a function that contain logarithms in it. For instance. $y = \log_{10} x$, the logarithm to base 10 is called the common logarithm. While the logarithm to base 2.718281... is called the natural logarithm. The natural logarithm represented by the symbol

ln. *Hence* $y = \log_{2.718281...} x = ln\ x$, in index form $e^x = y$. The value of e^x is called an exponential function. It is expressed as a Taylor's series represented as $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Example 1

In groups,

i. List the first four elements of the series $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

ii. List the first four elements of the derivative of the series $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

iii. What do you notice about $f(x) = e^x$ *and* $\frac{d}{dx}(e^x)$

Solution

i. $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \dots$

ii. $\frac{d}{dx}(e^x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 0 + 1 + \frac{x}{1} + \frac{x^2}{2!} + \dots \dots$

iii. $\frac{d}{dx}(e^x) = e^x$

In general if,

1. $f(x) = e^x$ then $f'(x) = e^x$
2. $f(x) = ln\ x,\ f'(x) = \frac{1}{x}$
3. $f(x) = a^x,\ f'(x) = a^x \ln x$

Exercise 3.3

Differentiate the following exponential and logarithmic functions in groups of three students.

1. $\ln(x^2 + 3x + 1)$
2. e^{3x^2}
3. $e^{x^2 + 3x}$
4. $\ln(2x^3 + 5x^2 - 3)$
5. e^{x^7}
6. 3^x

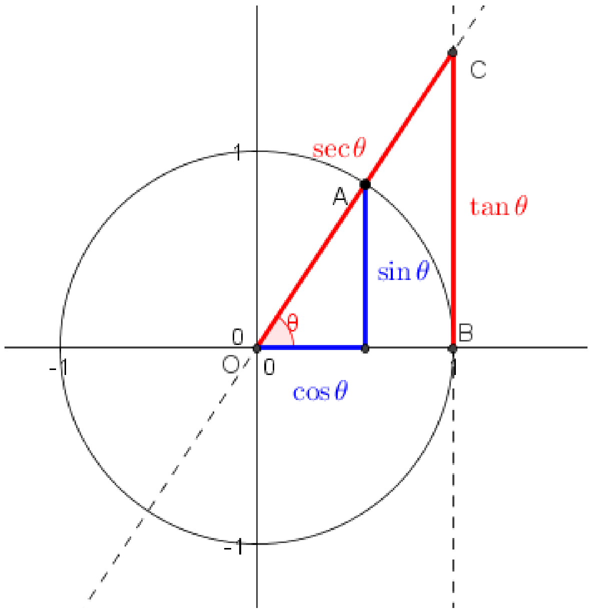
Derivatives of trigonometric functions

In the previous units trigonometric ratios have been discussed. They include ratio such as sin x, cos x, tan x, sec x, csc x, and cot x. Their relationship has also been discussed.

Some limits of trigonometric ratios have also been discussed on the unit of limits. One of this limit is $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ which is very useful in determining the derivative of trigonometric ratios using the first principal.

Example 1

In groups, study *figure 3.1* below for a unit circle drawn in radians and answer the questions that follow.



a. What is the relationship in magnitude of ,

i. θ , $\sin\theta$ *and* $1\ \tan\theta$

ii. $1, \cos\theta$ *and* $\frac{\sin\theta}{\theta}$

b. Why do we use radian in determining derivatives of trigonometric functions?

Solution

From the figure you notice,

$\sin\theta < \theta < 1\tan\theta.$

$\cos\theta < \frac{\sin\theta}{\theta} < 1.$

In determining derivative of trigonometric ratios using the first principle we rely on the two limits below in which x > 0 which is only measured in radian.

i.

$$\boxed{\lim_{\theta\to 0}\frac{sin\theta}{\theta} = 1 \quad .}$$

ii. $\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0 \quad .}$

From the definition of gradient by limit the derivative of *f(x)* is,

$$f'(\mathrm{x}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The limit definition of derivative, $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$ are used to derive derivative for trigonometric ratios.

Example 2

Prove that $\frac{d}{dx}(\sin x)$

Solution

$$f'(\mathrm{x}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(\sin x) = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h}$$

Using trigonometric identity $sin(x+h) = \sin x \ \cos h + \cos x \sin h$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \quad .$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \quad ..$$

$$= \sin \mathrm{x} \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \to 0} \frac{\sin \mathrm{h}}{\mathrm{h}} \quad ..$$

$$= (\sin x).0 + (\cos x)\,(1).$$

$$= \cos x.$$

Hence $\frac{d}{dx}\sin x = \cos x$

Example 3

Prove that $\frac{d}{dx}(\cos x) = -\sin x$

Solution

$$f'(\mathrm{x}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$\frac{d}{dx}(\cos x) = \lim_{x\to 0}\left(\frac{\cos((x+h) - \cos x}{h}\right)$. Using identity

$= \cos(\mathrm{x}+\mathrm{h}) = \cos\ \mathrm{x}\cos\mathrm{h} - \sin x\sin h$

$$= \lim_{h\to 0}\left(\frac{\cos\ \mathrm{x}\cos\mathrm{h} - \sin x\sin h - \cos x}{h}\right) .$$

$$= \lim_{h\to 0}\left(\frac{\cos\ \mathrm{x}\,(\cos\mathrm{h} - 1) - \sin x\sin h}{h}\right)$$

$$= cos\ x\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \sin x\ \lim_{h\to 0}\frac{\sin h}{h} .$$

Using the limits, $\lim_{\theta\to 0}\frac{sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

$$= cos\ x\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \sin x\ \lim_{h\to 0}\frac{\sin h}{h} = (\cos x).\,0 - (\sin x).\,1$$

$$= -\sin x.$$

Hence

$$= \frac{d}{dx}(\cos x) = -\sin x.$$

The other trigonometric derivatives are proved in a similar way to this. The table 3.1 below shows derivatives of trigonometric functions$f(x)$.

Activity 2

In groups of five students, prove the derivatives of the trigonometric functions in table 3.1 below.

$f(x)$	$f'(x)$
$sin\,x$	$cos\,x$
$cos\,x$	$-sin\,x$
$tan\,x$	$sec\,x\ \ tan\,x$
$cot\,x$	$-csc\,x^2$
$csc\,x$	$-csc\,x\,cot\,x$

Table 3.1

Example 4

i. Find the derivative of $\mathrm{y} = \sin 2x$.

ii. What do you notice about derivative of $\mathrm{y} = \sin ax$ and$\mathrm{y} = \cos ax$?

Solution

Let u = 2x

$$\frac{d}{dx} = 2.$$

$$\mathrm{y} = \sin u.$$

$$\frac{dy}{du} = \cos u.$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2\cos u = 2\cos 2x.$$

You note that:

$\frac{dy}{dx}(\sin ax) = a \cos ax$	$\frac{dy}{dx} \cos ax = -\sin ax$

Example 5

Find the derivative of $\mathrm{y} = \sin x \cos 3x$

Solution

Using the product rule

$If\ y{=}\ uv,\ y' = u'v + uv'$

$u = sin\, x.$

$\boxed{\mathrm{v} = \cos 3\mathrm{x}}$

$u' cos\, x.$

$\boxed{\mathrm{v}' = -3\sin 3\mathrm{x}}$

$y' sin\, x\; cos\, x - 3\, sin\, 3x\; cos\, 3x.$

Example 6

In pairs, find the derivative of the implicit equation $y^2 + \sin x + 2y = 0$

Solution

Using implicit differentiation

$$\frac{d}{dx}(y^2) + \frac{d(\sin x)}{dx} + \frac{d}{dx}2y = \frac{d}{dx}(0).$$

$$2y\frac{dy}{dx} + \cos x + \frac{2dy}{dx} = 0.$$

$$2y\frac{dy}{dx} + 2\frac{dy}{dx} = -\cos x.$$

$$\frac{dy}{dx} = \frac{-\cos x}{2y+2}.$$

Example 7

In pairs, find the derivative of $y = \frac{\sin x}{2x^2}$

Solution

Using the quotient rule

If $y = \frac{u}{v}$, $y' \frac{u^1v - uv^1}{v^2}$

$y = \sin x$, $u' = \cos x$

$v = 2x^2$, $v' = 4x$

$v^2 = 4x^4$

$$y' = \frac{2x^2 \cos x - 4x \sin x}{4x^4}$$

$$y' = \frac{2x(x\cos x - 2\sin x)}{4x^4}$$

$$y' = \frac{\cos x - 2\sin x}{2x^3}$$

Example 8

Differentiate $y^3 + sin\ x + y = 0$ with respect to x.

Solution

You notice the function is of form $g\ (x, y) = 0$.

Differentiate each term

$$\frac{d}{dx}y^2 + \frac{d}{dx}(\sin x) + \frac{d}{dx}(y) = \frac{d}{dx}0.$$

To differentiate $\frac{d}{dx}y^2$ we chain rule since y is a function of x. from chain rule you notice the derivative of y^n is derivative of = $(ay^n)^1 = \frac{dy}{dy}.\frac{dy}{dx}$ hence

$$\frac{d}{dx}(y^3) = \frac{dy}{dx}.\frac{dy}{dx} = 3y^2\frac{dy}{dx}.$$

$$\frac{d}{dx}y = \frac{dy}{dy}.\frac{dy}{dx} = 1\frac{dy}{dx}.$$

$$\frac{d}{dx}\sin x = \cos x.$$

Substituting back we get

$$3y^2\frac{dy}{dx} + \cos c + \frac{dy}{dx} = 0.$$

Making $\frac{dy}{dx}$ the subject we get

$$3y^2\frac{dy}{dx} + \frac{dy}{dx} = -\cos x.$$

$$\frac{dy}{dx}(3y^2 + 1) = -\cos x.$$

$$\frac{dy}{dx} = \frac{-\cos x}{3y^2+1}.$$

Exercise 3.3

Work in groups of five.

1. Prove the following trigonometric derivatives.
   a) $\frac{d}{dx}(\tan x) = \sec x \tan x$
   b) $\frac{d}{dx}(\sec x) = \sec x \tan x$
   c) $\frac{d}{dx}(\cot x) = -\csc x^2$
   d) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

2. Determine the derivative of each of the following functions
    a) $y = x^2 \sin x$,
    b) $y = 4\sin 3x$,
    c) $y = x^3 \cos x$
    d) $y = \sin x \cos 2x$
    e) $y = \frac{\sin x}{x+1}$
    f) $y = \frac{\sin 2x}{\cos 2x}$
    g) $y = (x+1)\sin 3x$
    h) $\sin(2x^2y^3) = 3x^3 + 1$
    i)
3. Figure 3.3 below shows a car moving on a flat surface heading to a tower 40m high with Patrick as an observer. If the angle of depression from P is θ and distance from p to the car is x.

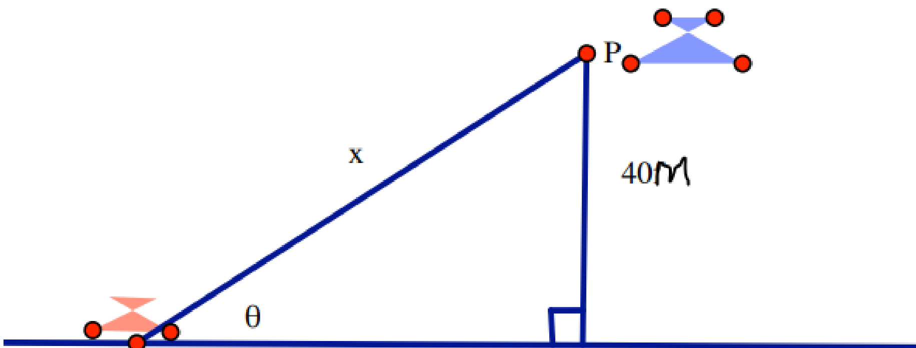


Figure 3.3

    a) Write an expression of x and θ.
    b) Write an expression for change in x with respect to θ
    c) Calculate how fast x change with respect to θ when $\theta = \frac{\pi}{4}$.

Integration

In the previous unit integration has been discussed as the reverse of differentiation. Integration by substitution, and integration by part to algebraic functions has been discussed. In this unit definite integrals will be discussed for area under curves. Integration of trigonometric functions, integration of powers of linear functions and their applications will be discussed.

Area under curves and integration

a) Area under a curve using vertical element $\Delta x = dx$ consider figure 3.4 below.

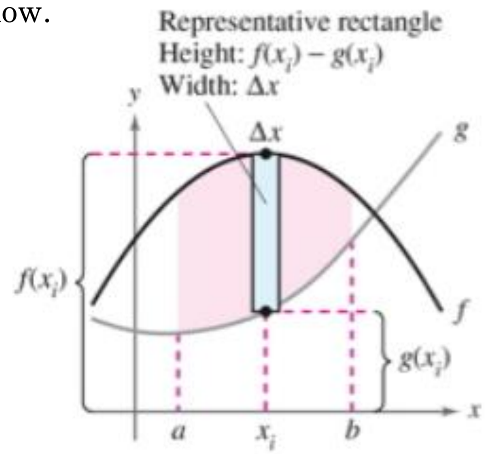


Figure 3.4

The area between the curve of function f and functioning g and line x= a and x = b can be calculated by subdividing there region into small rectangles of:-

With = Δx

Height = $f(x:) - g(xi)$

Area of I rectangle = $\Delta x(f(xi) - g(xi))$

If we subdivide the region (b -a) into n- rectangles then $i =$ $1,2,3, \dots \dots \dots \dots \dots . n.$

The approximated area will be the summation of the area of the n rectangle and hence

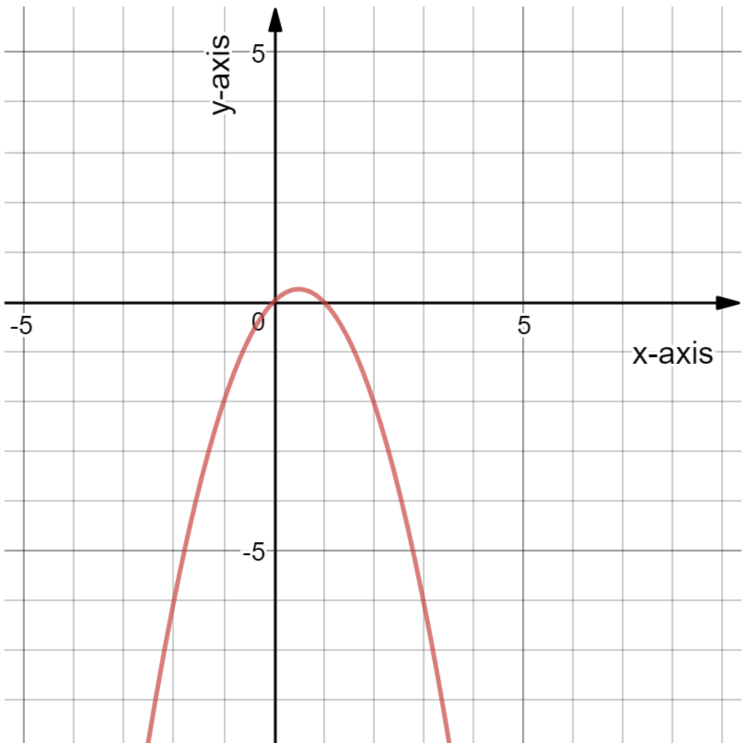
Area between f and g = $g = \sum_{i=n}^{n}(f(xi) - g(xi))\Delta x$

The higher the number of rectangles the more accurate is the area and hence,

$A = \lim_{n\to\infty} \sum_{i=1}^{n}(f(xi) - g(xi))\Delta x.$

Since $g(x)$ and $f(x)$ are continuous at (a, b) these limit exist and will be equal to the integral and hence,

(Area between f and g in interval (a, b) = $\int(f(x) - g(x)\,dx)$



In summary if function f and g are continuous on interval (a, b) and $f(x) > g(x)$ for all values in $[a, b]$ then the area of the region bounded by the graph of f and g and vertical line x =a and x = b is

(Area between f and g in against x in$[a, b]$) = $\int_a^b (f(x) - g(x))\, dx$

An integral with defined limits $[a, b]$ is called definite integrals since the theorem apply only at $f(x) \leq g(x) then$:

i. The area is always positive
ii. It is always advisable to sketch the graph and apply the functions appropriately. If the values of $f(x)$ and $g(x)$ interchange in the size between the intervals$[a, b]$ then the different regions area should be calculated differently.

To be more accurate when calculating the area under the curve (s) the steps followed include:-

1. Sketch the on the graph
2. Determine boundaries and set up the definite integral.
3. Integrate

Example 1

In groups of four students,

i. Use Desmos of otherwise draw the graph of $f(x) = 4x - x^2$
ii. Find the area bounded by $f(x) = 4x - x^2$, and x-axis.

Solution

The area bounded is shown in *Figure 3.5*

Figure 3.5

The intercepts are: at $x = 0, y = 0, (0,0)$

$$\text{At } y = 0, 4x - x^2 = 0$$

$$x(4 - x) = 0$$

$$x = 0 \text{ or } \quad 4 - x = 0$$

$$x = 4$$

Hence intercept are (0, 0) and (4 , 0)

$f'(x) = 4 - 2x$

$f'(x) = -2$ hence a maximum point exist at

4 – 2x = 0, x = 2, y = 8 – 4 = 4, (2 , 4)

The area between f and g against x = $\int_a^b (f(x) - g(x))\, dx$

$$f(x) = 4x - x^2$$

$$g(x) = 0$$

$$(f(x) - g(x)) = 4x - x^2$$

$a = 0, b = 4$

$\text{A} = \int_0^4 (4x - x^2)$

$\text{A} = \left(2x^2 - \frac{x^3}{3}\right)_0^4 \quad = \left[2.4^2 - \frac{4^3}{3}\right] - \left[2 \; x \; 0^2 - \frac{0^3}{3}\right] \ldots$

$= \frac{3^2}{3}$

$= 3\ \frac{2}{3}$ square units

Example 2

In groups of four students,

i. Use Desmos of otherwise draw the graph of $f(x) = x^2 - 2$ and $y = x$ on the same Cartesian plane.

ii. Calculate the area bounded by the curve $y = x^2 - 2$ and line $y = x$.

Solution

The graphs are shown in figure 3.6 below.

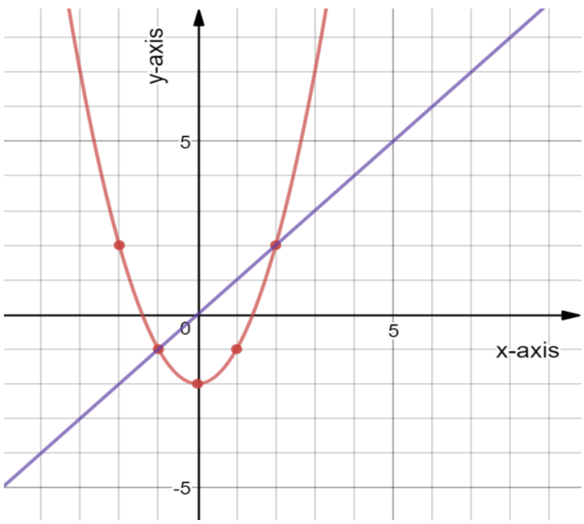


Figure 3.6

For area between f and g= $\int_a^b (f\ (x) - g(x))dx$ against Dx at { -a, b}

$$f(x) \ = \ x$$

$$g(x) = \ x^2 - 2$$

$$f\ (x) - \ g\ (x) \ = \ x - \ (x^2 - \ 2) \ = \ x - \ x^2 \ + \ 2 \ = \ -x^2 \ + \ x \ + \ 2$$

a= -1 , b= 2

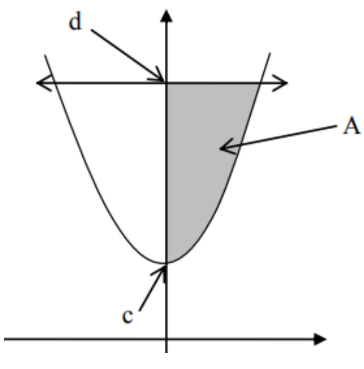
A= $\int_{-1}^{2}(-x^2 + x + 2)dx = \left[\frac{-x^3}{3} + \frac{x^2}{32} + 2x + c\right]_{-1}^{2}$

A = $\left(2 - \frac{8}{3} + \ 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) \ = \frac{9}{2}$

Area under a curve using horizontal element: $\boldsymbol{Dy}$ = dy

When area is calculated using the rectangles on horizontal sides. The values of their height can be expressed using the function of y, f(y)

Figure 3.7: Illustrate an area under a curve bounded $f(y)$. To calculate these area the horizontal element with typical rectangles of width dy is used as illustrated by *figure 3.8:*



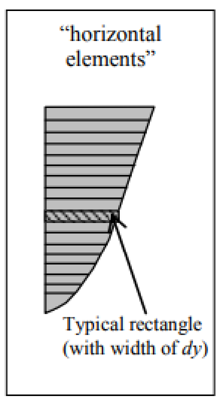


Figure 3.7 ***Figure 3.8***

The area calculated from horizontal components as from functions of y and is calculated as.

Area of between f(y) and g(y)= $\int_c^d (f(y) - g(y))dy$
Against y in interval {c, d}

The area is calculated like that in vertical component. $f(y) > g(y)$ through all {c, d} and the function $f(y)$ and $g(y)$ should be continuous at {c, d}

However, it is important to not most regions areas can be calculated by either vertical or horizontal components.

Activity 3

Calculate the area in the first quadrant and bounded by the function $y = x^2 + 2$, line $y = 4$ and $y = 2$.

***Solution*:**

The curve for function $y = x^2 + 2$ has appearance like that of figure 3.7 above. Hence

$y = x^2 + 2$

$x = \sqrt{y - 2}$.

For $A = \int_c^d (f(y) - g(y))dy$

$$f(y) = \sqrt{y-2}$$
$$g(y) = 0$$

$$f(y) - g(y) = \sqrt{y-2}$$

$C = 2$ and $d = 4$

$A = \int_2^4 \sqrt{y-2}\, dy = \frac{2}{3}\left(\sqrt{y-2}\right)^3 \Big|_2^4.$

$= \frac{2}{4}\left(\sqrt{4-2}\right)^3 - \frac{2}{3}\left(\sqrt{2-2}\right)^3 = \frac{2}{3}\left(\sqrt{2}\right)^3 - 0 = \frac{4\sqrt{2}}{3}$ square units

Exercise 3.4

Work in groups of four.

1. Sketch the graph of the area illustrated by functions in each of the following questions and calculate the described areas.
    a) Area enclosed by $y = x^3$ and $y = 3\sqrt{x}$
    b) Area enclosed by $y = x^3 + 1$ and $y = (x+1)^2$
    c) Area enclosed by $y = \cos x$, $y = x$ and y-axis and is in first quadrant.
    d) Area bounded by $y = x^3$ and $y = x^2$
    e) Area bounded by $y = 3x^3 - x$ and $y = x^3 + x$
    f) Area bounded by $x^2 = y$ and $x = y + 2$

2. Find the area bounded by the curve $y = x^2 - 4$. And lines $y = 0$, and $x = 4$.

3. Find the area of the region bounded by curve $x = y^2 - 2$ and $x = y$.

4. Draw a detailed sketch to illustrate the region enclosed by line $y = x + 6$, $x = 0$, $x = 5$ and $y = x^2$ and hence calculate these area.
    a) Calculate the area of region bounded by the following function by integrating with respect to x
        i. $y = 0$, $x = 3$ and $y = 3x^2$
        ii. $x = 4 + y^2$ and $2x = y^2$

b) Calculate the areas in (a) above by integrating with respect to y.

c) What do you notice?

5. Draw a graph to show area bounded by $y = x^2, y = 2 - x, x \geq 0$ and $x = 0$.
Determine the area bounded as described in (a) above.

Integration of power of linear functions (integration by substitution)

A linear function is a function of the form $ax + b$ where a and b are constants and $a \neq 0$. Integration of powers of linear functions hence involve determining definite and indefinite integrals of the functions of the form of $\int (ax + b)^2\, dx$.

A method of substitution occasionally performs apparently easier integration. This method is called method of substitution. Where the linear function becomes the u, hence

$$u = ax + b$$

As discussed in differentiation, a method of u-substitution reduces a composite function to a single function. It can be used to solve problem of the nature of

i. The integral of power of linear functions $\int (ax + b)^2\, dx$

ii. Integral of composite function made of product of a composite function $f(g(x))$ and derivative of the inner function $g'(x)$ hence

$$\int f\big(g(x)\big)\, g(x) dx$$

Example 1

Evaluate the definite integral $\int_0^1 (3x + 4)^4 \mathrm{dx}$

Solution

By substitution

Let u= 3x + 4, $\frac{du}{dx} = 1$, du = 3dx, dx =$\frac{1}{3}$du.

$$\int_{x=0}^{x=1} (3x + 4)^4 dx = \int_{x=0}^{x=1} u^4 \frac{1}{3} du = \frac{1}{3}\int_{u=4}^{u=7} u^4\, du = \frac{1}{3}\left[\frac{1}{5}u^5 + c\right]_4^7$$

Not at x=4, u = 4, at x = 1, u = 7

$= \frac{1}{15}(u)^5 + |_4^7$

$= \frac{1}{15}[7^5 - 4^5] = \frac{1}{15}[7^5 - 4^5] = 1052.2$

Example 2

Evaluate $\int \frac{4x}{\sqrt{2x^2+1}} dx$

Solution

By substitution

U=2x2 +1 ,$\frac{du}{dx} = 4x$, dx =$\frac{du}{4x}$

$$\int 4x.\frac{1}{u^{\frac{1}{2}}}\frac{du}{4x} = \int u^{-\frac{1}{2}} du = \frac{1}{2}u^{\frac{1}{2}} + c$$

$$= \frac{1}{2}(2x^2 + 1)^{\frac{1}{2}} + c$$

$$= \frac{1}{2}\sqrt{2x^2 + 1} + c$$

Integral for exponential and logarithmic functions

Further reading activity.

In groups, prove that

i. $\int \frac{1}{x}\, dx = \ln|x| + c$

ii. $\int e^x dx = e^x + c$

iii. $\int a^x dx = \frac{a^x}{\ln|2|} + c$

Example 1

Evaluate $\int 2^x dx$

Solution

Consider, $\int a^x dx = \frac{a^x}{\ln|a|} + c$, applying this $a = 2$

$$\int 2^x dx = \frac{2^x}{\ln|2|} + c$$

Example 2

In pairs. Simplify $\int \frac{3}{2x+1}\, dx$

Solution

Let, $u = 2x + 1$, $\frac{du}{dx} = 2$, $dx = ½\, du$.

$$\int \frac{3}{2x+1}\, dx = \int \frac{3}{2u}\, du = \frac{3}{2} \int u\, du$$

$= \frac{3}{2} \ln u + c$

$= \frac{3}{2} \ln|2x + 1| + c$

Exercise 3.5

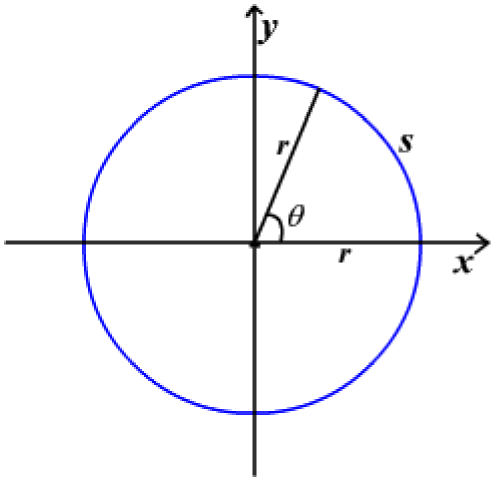
Work in pairs.

Evaluate the following indefinite and definite integral in question 1-6.

1. $\int (x+4)^5 \text{dx}$
2. $\int_1^3 (9+x)^2 \text{ dx}$
3. $\int (y-2)^3 \text{dy}$
4. $\int (y+5)^4 \text{dy}$
5. $\int (2x-1)^? \text{dx}$
6. $\int_{-1}^{1} (1-x)^3 \text{ dx}$
7. By use of u-substitution find the integral of
   a) $\int 5x\,(\sqrt{1-x^2})\,dx$
   b) $\int \frac{x^3}{\sqrt{x^4+16}} dx$
   c) $\int_0^1 \frac{x^3}{\sqrt{x^4+12}} dx$
   d) $\int 5x^2\sqrt{1-x^3}\,dx$
8. Determine the area bounded by the curve of f(x) = $x^3\sqrt{x^4+1}$ line x=5 and y-axis.
9. Integrate the following logarithmic and exponential functions in groups of four.
   a. $\int \frac{3}{6x+1}\,dx$
   b. $\int \frac{x}{6x+2}\,dx$
   c. $\int e^{3x}\,dx$
   d. $\int 5^{(4x+2)}\,dx$

Integration of trigonometric functions

Previously integration has being described as the reverse process of differentiation. The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. One way to measure angles is in radians. To define a radian, use a central angle of a circle (an angle whose vertex is the center of the circle). One radian is the measure of a central angle that intercepts an arc ***s*** equal in length to the radius **r** of the circle.



The derivative of cosine and sine has also been discussed as follows:-

$f(x)$	$f'(x)$
Sin x	Cos x
Cos x	-Sin x

The knowledge on inverse of functions ids used to determine the integral of trigonometric functions.

Example 1

i. Using Desmos or otherwise draw the graphs of
   a. y= Sin x in the interval $-2\Pi \leq x \leq 2\Pi$
   b. y=Cos x in the interval $-2\Pi \leq x \leq 2\Pi$

ii. Does the inverse of y= Sin x and y=Cos x exist in the interval$-2\Pi \leq x \leq 2\Pi$. If no at which interval does it exist?

Solution

Consider *figure 3.9* and *3.10* below for the graphs of y= Sin x and y=Cos x respectively.

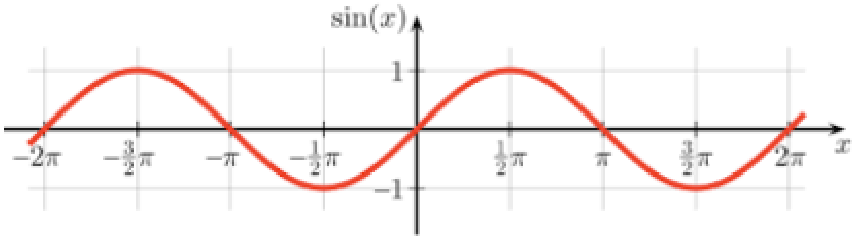


Figure 3.9

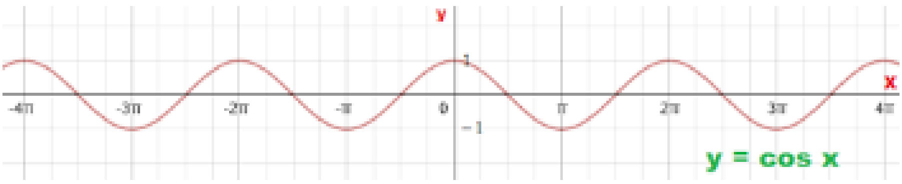


Figure 3.10

You notice that the graph y= Cos x and y= Sin x are functions since for each domain there is one range. It is a one to one mapping. The inverse of these function you notice they do not exist since for each domain of inverse the range will have several ranges. The inverse of y= Cos x and y= Sin x does not hence exist.

There is clearly illustrated by the reflection of these functions on line y=x

On restricting the domain of the range you notice on restricted domain the inverse exist for instance inverse of $y=$ - $Sin\ x$ exist for $sin^{-1} x = y\ at\ \frac{-1}{2}\pi < x \leq \frac{1}{2}\pi$ and inverse of $y = \ Cos\ x$ defined as $\cos^{-1} x = y$ exist at $0 \leq\ x\ \leq\ \pi$.

Since integration is the reverse of differentiation and the inverse of $sin\ x = y$ and $cos\ x = y$ exist on restricted domain we can hence prove that the integral of

$f(x) = \sin x$ and $f(x) = \cos x$ are the reverse of their differentiation. Hence,

$f(x)$	$\int f(x)\mathrm{dx}$
$Sin\ x$	$-Cos\ x$
$Cos\ x$	$Sin\ x$

Using these two integrals it is possible to derive integrals of other trigonometric functions. In solving the trigonometric integral trigonometric identities discussed in unit trigonometry are used.

Table 3.1: Below shows the useful trigonometry identities.

Table 3.1

TRIGONOMETRIC IDENTITIES	
RECIPROCAL IDENTITIES $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$	**DOUBLE-ANGLE IDENTITIES** $\sin 2x = 2\sin x\cos x = \frac{2\tan x}{1+\tan^2 x}$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$ $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$
PYTHAGOREAN IDENTITIES $\sin^2 x + \cos^2 x = 1$	**HALF-ANGLE IDENTITIES**

$\sin^2 x = 1 - \cos^2 x$ $1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$	$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$ $\tan\frac{x}{2}$ $= \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
SUM AND DIFFERENCE IDENTITIES $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	

In addition, since integral is reverse of differentiation the following derivatives of common trigonometric functions are also useful. Table 3.2 below shows derivative of common functions.

Table 3.2

$$\frac{d}{dx}(\sin(x)) = \cos(x)\ \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)\ \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)\ \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

Using the above two integrals and the identities it is possible to derive the integral of other trigonometric functions.

Example 2

Prove that $\int \tan x\, dx = -\mathrm{In}|\cos x| + c$

Solution

From identity tan x = $\frac{\sin x}{\cos x}$ $\int$ tan x dx = $\frac{\sin x}{\cos x}$ dx

$u = \ cos\ x, \frac{du}{dx} = -\sin x$, dx =$\frac{du}{-\sin x}$.

Substituting basis the value of dx and cos x=u

$$\int \frac{\sin x}{\cos x} dx = \int \sin x \,.\frac{du}{\sin x} = \int \frac{1}{-u} du = -\int \frac{1}{u} du$$

$$= \ -In|u| \ + \ c$$

$$= \ -In|cos\ x| + \ c$$

Example 3

Prove that $\int \sec x \ dx = In|\sec x + \tan x| + c$

Solution

To enable us make substitution multiply both numerator and denominator by $(sec\ x \ + \ tan\ x)$

Hence,

$$\int \sec x \ dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

Using u-substitution $u = \ sec\ x \ + \ tan\ x$

$$\frac{du}{dx} = \sec x \tan x + sec^2\, x = \sec x(\tan x + \sec x)$$

$$dx = \frac{du}{\sec x(\tan x + \sec x)}$$

Substituting back the u and dx

$$\int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec x\,.\,(\sec x + \tan x).\,du}{\sec x\,(\tan x + \tan x)\,.\,u} = \int \frac{du}{u}$$

$= \text{In} \ + c$, substituting u.

$= \text{In} \,|\, \sec x + \tan x \,|\, + c$

Table 3.3 below shows the integral of common trigonometric functions.

Table 3.3

$$\int \cos x \;\, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \;\, dx = \tan x + C$$

$$\int \csc x \cot x \, dx = \; -\csc x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

Since derivatives and integrals are inverse of each other's and are concurrently used in evaluating integrals it is important to master the derivatives and integral of common trigonometric tables. Since one is always required to apply them. *Table 3.4* below shows the derivatives and integral of these functions.

Table 3.4

Derivatives or differentiation Formulas	Antiderivatives or Integration Formulas
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}[\cos x] = \sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}[\cot x] = \csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = \csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

To evaluate the integral of trigonometric functions different methods of differentiations and integration discussed previously are applied as illustrated in the example and exercises that follows.

Example 4

Integrate $\int \sin 3x\, dx$

Solution

$u = 3x, \frac{du}{dx} = 3, \text{dx} = \frac{1}{3} du$

And $\int \sin 3x\, dx = -\frac{1}{3} \cos 3x + c$

Example 5

In pairs, integrate $\int (\sin x + \cos x)^2 \text{dx}$

Solution

And $\int(\sin x + \cos x)^2 dx = \int \sin^2 x + 2\sin x \cos x + \cos^2 x)dx$
But $\cos^2 x + \sin^2 x = 1$, hence
So $\int(2\cos x \sin x + 1) \quad dx = 2\int \cos x \sin x \, dx + \int 1 \quad dx$
$= x + 2 \, 2\int \cos x \sin x \, dx$
By substitution $u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{du}{\cos x}$
Hence,

$$\int(2\cos x \sin x + 1) \quad dx = x + 2\int \cos x \,.\, u . \frac{du}{\cos x} = x + 2\int u \quad du$$
$$= x + u^2 + c$$
$$= x + sin^2 x + c$$

Example 6

In pairs, integrate $\int 3\cos^2 5x \, dx$.

Solution

$\int 3\cos^2 5x \, dx = 3\int \cos^2 5x \, dx$.

Using double angle identity formula 2 cos2θ= 1 + cos2θ we obtain

$3\int \frac{1+\cos 2\,(5x)}{2} dx = \frac{3}{2}\,(1 + \cos 10x)dx$.

$= \frac{3}{2}(x + \int \cos 10x \, dx)$

$= \frac{3}{2}x + \frac{3}{12}\int \cos 10x \, dx$.

Using $u = 10x$, $\frac{du}{dx} = 10$, $dx = \frac{du}{10}$ and hence

$= \frac{3}{2}x + \frac{3}{12}\int \cos u \,.\frac{du}{10} = \frac{3}{2}x + \frac{3}{20}\sin u + c$.

$=\frac{3}{2}x + \frac{3}{20}\sin 10x + c$

Exercise 3.6

Work in groups.

1. Evaluate $\int \cos(3x+4)\,dx$
2. Evaluate $\int \sin(7x+4)\,dx$
3. Evaluate $\int_0^{\frac{\pi}{2}} \cos(1-x)\,dx$
4. Evaluate $\int \frac{1}{5x}(\sin\sqrt{x})\,dx$
5. Evaluate $\int -2x\sin\,(1-x^2)\,dx$
6. Evaluate $\int \frac{\cos x}{1+sinx}dx$
7. Evaluate $\int x\sin(2x^2)\,dx$
8. Evaluate $\int \frac{\cos x}{(5+\sin x)^2}dx$
9. Evaluate $\int \sin\,3x\,dx$
10. Integrate $\int \frac{cos^2x}{1+\sin x}dx$
11. Evaluate $\int \frac{\cos 5x}{3+\sin 5x}dx$
12. Evaluate $\int 5\sec 4x\tan 4x\;\,dx$

Application of integration to kinematics

During the differentiation we learnt that velocity is differentiated function of displacement, and acceleration is the differential function of velocity. Reversing this process we conclude that velocity (v) is the integration of acceleration (a) and displacement (s) is integration of velocity (v) in respect to t. and hence

1. $\int adt = v$, and$\int vdt = s$

Example

A car passed a bump at an acceleration of 5m/s² as shown in fig 3.11 below. After 2 seconds from the bump it was moving at a speed of 30m/s.

Figure 3.11

a) Write an expression for velocity of the car.
b) If the car had covered 100m from a point A to where the bump was determine
   i. An expression for the distance (s) in term of t (time after the bump)
   ii. The distance covered by the car in 10 seconds from passing the bump.

Solution

a. a= 5m/s
   v=$\int adt = \int 5dt$ = 5t + c
   v = 5t + c since at t = 2, v = 30
   30 = 5 x 2 +c and c = 20
   Hence v = 5t + 20

b. i) $s = \int vdt = \int 5t + 20dt = \frac{5}{2}t^2 + 20t + c$
   s = $\frac{5}{2}t^2 + 20t + c$
   Since at t = 0, s = 100
   s = 0 + 0 +c, c =100
   $s = \frac{5}{2}t^2 + 20t + 100.$
   ii) $s(10) = \frac{5}{2}x\ 10^2 + 20x10 + 100$

$s = 250 + 200 + 100$
$s = 550m$

Exercise 3.7

Work in pairs.

4. a) A ball is thrown downward from a height 512m with a velocity at 64m/s as shown in fig 3.12. How long will it take to reach the ground?

Figure 3.12

b) What is the velocity of the ball when it hits the ground?

2. A falling rock is accelerating at a rate of $4m/s^2$ from a hole 35m below the ground level. How high above the ground will it be after 6 seconds?

UNIT 4

PARTIAL FRACTIONS

Introduction

A fraction is a rational number with both numerator and denominator. In this unit, we will discuss method of expressing quotients of polynomials and algebraic expression in terms of two or more fractions.

Just like one can write other fractions in a different form the polynomials also can be expressed in parts.

For instance the fraction $\frac{3}{5}$ can be written in partial fractions $\frac{3}{5} = \frac{2}{5} + \frac{1}{5}$ However, in this unit we will consider algebraic fraction such as,

$f(x) = \frac{g(x)}{h(x)}$ where, $g(x) < h(x)$. This is indicated by larger power in $h(x)$.

Example 1

In pairs, simplify the expression $\frac{x^3}{x^2-1}$ into a mixed number.

Solution

Since the numerator has bigger power than the denominator the fraction has been decomposed into parts

Consider $\frac{x^3}{x^2-1} = \frac{x(x^2-1)+x}{x^2-1}$

You notice x^3 can be written as $x(x^2-1)+x$ while on expansion the added term is eliminated technically $x(x^2-1) - x = x^3 - x + x = x^3$

To Simplify $\frac{x^3}{x^2-1} = \frac{x(x^2-1)+x}{x^2-1} = \frac{x(x^2-1)}{x^2-1} + \frac{x}{x^2-1} = x + \frac{x}{x^2-1}$

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

The decomposition of partial fraction generate fractions that can be integrated by method of substitution. The decomposition depends on the nature of denominator and hence it's important to divide improper fractions.

Example 2

In pairs, simplify the improper fraction below.

$$\frac{x^4+3x^3-4x^2}{x^2+3x-4}$$

Solution

Express the denominitor as a product of two factors and then simplify,

$$\frac{x^2(x^2+3x-4)}{x^2+3x-4} = x^2$$

Exercise 4.1

In pairs, Simplify:

1 $\frac{x^3+6x^2+13x-12}{3x+3}$

2 $\frac{x^4-23x^3+49x+4}{x^2+x-3}$

3 $\frac{3x^2-11x^2-4}{x-4}$

4 $\frac{6x^2+7x+6}{x-4}$

5 $\frac{6x^2y^2-10x^2y}{4y}$

Partial Fractions with linear factor(s) in the denominators

A fraction with linear factor denominator is a fraction whose denominator has expression of the form $(ax+b)^n$ where a and b are integers and $a \neq 0$ and $n \neq 0$. When n>1 the denominator is said to have more than one factor. The denominator may also have different linear factors such as

$(ax+b)(cx+d)$ and many others. The fractions with linear or linear factor denominator of the general cases above are decomposed into partial fractions in this unit.

1. $\frac{A}{ax+b}$ Need not to be decomposed. For example , $\frac{3}{(x+3)}$

2. $\frac{g(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{(cx+d)}$ **Linear different factor** format in denominator, for example $\frac{2x}{(x+3)(3x+1)}$

3. $\frac{g(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$ **Linear repeated factors** format in denominator. For example $\frac{2x-1}{(x-5)^2}$

4. $\frac{g(x)}{(ax+b)^3} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{c}{(ax+b)^3}$ **Linear repeated factors** format in denominator. For example, $\frac{2x}{(x-3)^{23}}$

5. $\frac{g(x)}{(ax+b)^3(cx+d)} = \frac{A}{cx+d} + \frac{B}{ax+b} + \frac{C}{(ax+b)^2} + \frac{D}{(ax+b)^3}$ **Linear and linear repeated factors** format in denominator. For example , $\frac{x^2-2}{(x+3)^3(x-2)}$

A repeated factor has as many parts as its factors. A, B , C …… are integers that can be solved by simplifying the R.H.S and L.H.S (Right hand side and left hand side) of the equation of the fraction to whole numbers then solve for A, B and C by either of the methods below:

I. Comparing similar coefficients
II. Substituting x that eliminate one of A,B or C by a zero product

Example 1

In groups, express $\frac{2x}{(x+3)(3x+1)}$ into partial fraction.

Solution

The fraction has repeated factor format of case 2 above i.e.

$$\frac{g(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{(cx+d)}$$

Applying to the expression we get

$$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{(3x+1)}$$

To simplify this, multiply each term by the L.C.M to eliminate fractions

L.C.M = $(x+3)\,(3x+1)$

$$\frac{2x(\mathrm{x}+3)\,(3\mathrm{x}+1)}{(x+3)(3x+1)} = \frac{A(\mathrm{x}+3)\,(3\mathrm{x}+1)}{x+3} + \frac{B\,(\mathrm{x}+3)\,(3\mathrm{x}+1)}{(3x+1)}$$

After cancellation we get.

$$2x = A(3x+1) + B(x+3)$$

These equation can be solved for A and B using two different ways. They include;

1. Expanding and calculating coefficient
2. Substituting x that eliminate A or B by zero factor product

Expanding and calculating coefficient

$$2x = A(3x+1) + B(x+3)$$

$$2x = 3Ax + A + Bx + 3B$$

Coefficient of x^2 on L.H.S equal to coefficient of x^2 of RHS.

Coefficient of x on LHS equal to coefficient of x in RHS.

Constant on R.H.S =Constant on L.H.S hence:

$$2 = 3A + B \ldots..1$$

$$0 = \ A + 3B \ldots..2$$

Solving the simultaneous equation

3A+B=2.....1

3A+9B........2

-8B=2

$B = \frac{-1}{4}$, $A = -3B = \ -3 \times \frac{-1}{4} = \frac{3}{4}$

$$A = +3/4B$$

Hence, $A = \frac{3}{4}$, $B = -\ \frac{1}{4}$ substituting back to the main equation we get.

$$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{(3x+1)}$$

$\frac{2x}{(x+3)(3x+1)} = \frac{3/4}{x+3} + \frac{-1/4}{(3x+1)}$ or

$$\frac{2x}{(x+3)(3x+1)} = \frac{3}{4(x+3)} - \frac{1}{4(3x+1)}$$

Alternatively,

The equation, $2x(3x+1) = A(3x+1) + B(x+3)$ Can also be solved by substituting x that eliminate A or B by zero factor product.

Substituting x that eliminate A or B by zero factor product.

To eliminate A use, $A[3x+1] = A \times 0$

Hence, $3x + 1 = 0$

$$x = -\frac{1}{3}$$

Substituting x=-1/3 in the equation we get

$$2x(3x+1) = A(3x+1) + B(x+3)$$

$$2(-1/3)\ (3x - \frac{1}{3} + 1) = A(3x - \frac{1}{3} + 1) + B(-\frac{1}{3} + 3)$$

$$2(-\frac{1}{3}).0 = A.O + B$$

$$B = -\frac{1}{4}$$

To eliminate B use x = -3

$$3[x][3x+1] = A[3x+1] + B[x-3]$$

$$3[-3][3[-3] + 1 = A[3[-3] + 1]$$

$$A = 3/4$$

Hence

$$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{(3x+1)}$$

$\frac{2x}{(x+3)(3x+1)} = \frac{3/4}{x+3} + \frac{-1/4}{(3x+1)}$or

$$\frac{2x}{(x+3)(3x+1)} = \frac{3}{4(x+3)} - \frac{1}{4(3x+1)}$$

The method of substituting x that eliminate one of the constant is preferred since equating coefficient may result into simultaneous equations that are tedious to solve

Example 2

In groups, express $\frac{2x-1}{(x-5)^2}$ into partial fractions.

Solution

You notice the equation is of the form

$\frac{g(x)}{(ax-b)^2} = \frac{A}{ax-b} + \frac{B}{(ax-b)^2}$**Linearrepeated factors** format in denominator

Substituting on it

$$\frac{2x-1}{(x-5)^2} = \frac{A}{(x-5)} + \frac{B}{(x-5)^2}$$

Multiplying by LCM, L.C.M =(X-5)(x-5) in each term

$$\frac{(2x-1)(\mathrm{x}-5)(\mathrm{x}-5)}{(x-5)^2} = \frac{A(\mathrm{x}-5)(\mathrm{x}-5)}{(x-5)} + \frac{B(\mathrm{x}-5)(\mathrm{x}-5)}{(x-5)^2}$$

After cancellation we obtain,

$$2x-1 \;=\; A(x-5)+B$$

Eliminate A by substituting x=5

$$2\times 5 - 1 \;=\; B$$

$$B \;=\; 9$$

Substituting B=9 and x =0 to 2x-1 = A(x-5)+B we get

$$2 \times 0 - 1 = A(0-5) + 9$$

$$-1 = -5A + 9$$

$$5A = 10, A = 2 \qquad we\ get\ A = 2\ and\ B = 9$$

Substituting back A=2 and B=9 to the main equation we get.

$$\frac{2x-1}{(x-5)^2} = \frac{A}{(x-5)} + \frac{B}{(x-5)^2}$$

$$\frac{2x-1}{(x-5)^2} = \frac{2}{(x-5)} + \frac{9}{(x-5)^2}$$

Example 3

In pairs, find the decomposition of $\frac{x^2-2}{(x+3)^3(x-2)}$ into partial fraction.

Solution

You notice the denominator has linear and linear repeated denominator and hence decompose to the general format of

$$\frac{g(x)}{(ax+b)^3(cx+d)} = \frac{A}{cx+d} + \frac{B}{ax+b} + \frac{C}{(ax+b)^2} + \frac{D}{(ax+b)^3}$$

Substituting to this equation we get.

$$\frac{(x^2-2)}{(x+3)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)^3}$$

Multiplying each term by the LCM,

$LCM = (x+3)^3(x-2)$

$$\frac{(x^2-2)(x+3)^3(x-2)}{(x+3)^3(x-2)}$$
$$= \frac{A(x+3)^3(x-2)}{x-2} + \frac{B(x+3)^3(x-2)}{x+3} + \frac{C(x+3)^3(x-2)}{(x+3)^2} + \frac{D(x+3)^3(x-2)}{(x+3)^3}$$

After cancellation we get,

$$(\boldsymbol{x^2-2}) = \boldsymbol{A(x+3)^3 + B(x+3)^2(x-2) + C(x+3)\ \ (x-2) + \mathrm{D}(\mathrm{x}-2)}$$

This is the equation we use to solve foe the coefficients.

By substituting, $\mathrm{x} = -3$, $(x+3)^2 = 0$, $(x+3)^2 = 0$, $(x+3) = o$,

Hence,

$$9-2 = 0+0+0+D(-5)$$

$$7 = -5D$$

$$7 = -5D$$

Substituting $x = 2$, in

$$(\boldsymbol{x^2-2}) = \boldsymbol{A(x+3)^3 + B(x+3)^2(x-2) + C(x+3)\ \ (x-2) + \mathrm{D}(\mathrm{x}-2)}$$

$(x-2) = 0$ and hence all terms $(x-2)$ with will be reduced to zero.

$$4-2 = 125A$$

$$A = \frac{2}{125}$$

Substituting $x=1$ in,

$$(\boldsymbol{x^2-2}) = \boldsymbol{A(x+3)^3 + B(x+3)^2(x-2) + C(x+3)\ \ (x-2) + \mathrm{D}(\mathrm{x}-2)}$$

We get.

$-1 = 64A + 4B - 4C - D$ substituting, $A = \frac{2}{125}$ and $D = \frac{-7}{5}$ we get

$$4B + 4C = \frac{47}{125} \;\ldots\ldots 1$$

Substituting *x=0* in

$(\boldsymbol{x^2 - 2}) = \boldsymbol{A}(\boldsymbol{x+3})^{\mathbf{3}} + \boldsymbol{B}(\boldsymbol{x+3})^{\mathbf{2}}(\boldsymbol{x-2}) + \boldsymbol{C}(\boldsymbol{x+3}) \;\; (\boldsymbol{x-2}) + \mathbf{D}(\mathbf{x-2})$

We get,

$$-2 = 27A - 18B - 6C - 2D$$

$$18B - 6C = \frac{-654}{125} \;\ldots\ldots\; 2$$

We form simultaneous equations.

$4B + 4C = \dfrac{47}{125} \;\ldots\ldots 1$

$18B - 6C = \frac{-654}{125} \;\ldots\ldots\; 2$

On solving the equation we get

$$B = \frac{-2}{125}, \quad C = \frac{-23}{25}$$

Therefore, $\quad A = \frac{-2}{125}\,,\; B = \frac{2}{125}, \quad C = \frac{23}{25}$ and $D = \frac{-7}{5}$

Substituting the constants to the main equation we get

$$\frac{(x^2-2)}{(x+3)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)^3}$$

$$\frac{(x^2-2)}{(x+3)^3(x-2)} = \frac{\frac{2}{125}}{x-2} + \frac{\frac{-2}{125}}{x+3} + \frac{\frac{23}{25}}{(x+3)^2} + \frac{\frac{-7}{5}}{(x+3)^3}$$

Or

$$\frac{(x^2-2)}{(x+3)^3(x-2)} = \frac{2}{125(x-2)} - \frac{2}{125(x+3)} + \frac{23}{25(x+3)^2} - \frac{7}{5(x+3)^3}$$

Exercise 4.2

To be done in groups.

Express each of the following fractions in partial fractions.

1 $\frac{(x^2-x)}{(x-3)(x+5)\quad(x-1)}$

2 $\frac{(x\quad+3)}{(x-1)^3}$

3 $\frac{(2x\quad-1)}{(x+2)^2(x-3)}$

4 $\frac{(3x^2+1)}{(x-5)^2(x+2)}$

5 $\frac{(x^2+7x)}{(x-1)\quad(x+2)(x-4)}$

6 $\frac{(6x^2-3x)}{(x-2)\quad(x+2)}$

Partial Fractions of Functions with Quadratic Expression in the Denominator

A quadratic expression is an expression of the form of $ax^2 + bx + c$ where a, b and c are integers

$a\ =\ 0$and$x$ is variable. A quadratic expression with real integer roots can be factorized into the forms:

1. $(a\ + b)(a\ + b)$ similar to $(a\ + b)^2$
2. $(a - b)(a - b)$ similar to $(a - b)^2$
3. $(a\ + b)(a - b)$ similar to $(a^2 - b^2)$

Where a is a variable and b are constant. Other quadratic expressions that cannot be factorized remain in the general formof $ax^2 + bx + c$.

On decomposing algebraic fractions into partial fractions with quadratic expression(s) with factors in the denominator of the nature in the first 3 cases we use their linear factor on denominator. The use of factors reduces the decomposition from quadratic to linear factors. The decomposition used in this instances are similar to the previous decomposition in the previous part. This include;

1. $\frac{g(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{(cx+d)}$**Linear different factor** format in denominator

2. $\frac{g(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$ **Linear repeated factors** format in denominator

3. $\frac{g(x)}{(ax-b)^2} = \frac{A}{ax-b} + \frac{B}{(ax-b)^2}$ **Linearrepeated factors** format in denominator

In case the denominator has a quadratic denominator that cannot be factorized we use decomposition below. Such a denomination is called use of i**rreducible denominator.** Quadratic Expression Decomposition with Irreducible Denominators are;

4. $\frac{g(x)}{ax^2+bx+c} = \frac{Ax+B}{ax^2+bx+c}$**Irreducible quadratic** denominator

5. $\frac{g(x)}{(ax^2+bx+c)^2} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$**repeated Irreducible quadratic** denominator

If the denominator has a product of linear and quadratic expression, a combination of the two formats are applied. For instance,

6. $\frac{g(x)}{(ax^2+bx+c)(dx+ex)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$**linear andIrreducible quadratic** denominator

Example 1

In pairs, express $\frac{2x-1}{x^2-10x+25}$ in partial fractions.

Solution

We check if the denominator can be reduced to linear format by factorization.

Factorize $x^2 - 10x + 25$

$$P + q = -10, \ \ Pq = 25$$

$P = -5, q = -5$ Since the factors exist we factories the quadratic expression and reduce it to linear factors as follows;

$$x^2 - 5x - 5x + 25$$

$$x(x-5) - (5x-5)$$

$$(x-5)(x-5) = (x-5)^2$$

Hence,

$$\frac{2x-1}{x^2-10x+25} = \frac{2x-1}{(x-5)^2}$$

This fraction expression decompose in general form of

$\frac{g(x)}{(ax-b)^2} = \frac{A}{ax-b} + \frac{B}{(ax-b)^2}$ **Linear repeated factors** case in denominator

Substituting to the general expression we get

$$\frac{2x-1}{(x-5)^2} = \frac{A}{(x-5)} + \frac{B}{((x-5))^2}$$

Multiply through by L.C.M, $LCM = (x-5)^2 = (x-5)(x-5)$

$$\frac{(2x-1)(x-5)(x-5)}{(x-5)^2} = \frac{A(x-5)(x-5)}{(x-5)} + \frac{B(x-5)(x-5)}{\left((x-5)\right)^2}$$

After cancellation we get

$(2x-1) = A(x-5)$ +B……… 1

Substituting x =5 to eliminate A in equation 1 as shown below

$$2 \times 5 - 1 \ = \ A(5-5) \ + \ B$$

$$9 \ = \ A \times 0 \ + B$$

$$B = \ 9$$

Substituting A=0 and B=9 to equation 1 we get

$$2 \times 0 \ - 1 \ = \ A(\ 0 - 5) \ + 9$$

$$5A = 10 \ \ and \ A = 2$$

Hence, $A = 2$ and $B = 9$

Substituting $A = 2$ and $B = 9$ to the main equation

$\frac{2x-1}{(x-5)^2} = \frac{A}{(x-5)} + \frac{B}{((x-5))^2}$ we get

$$\frac{2x-1}{(x-5)^2} = \frac{2}{(x-5)} + \frac{9}{(x-5)^2}$$

Example 2

In groups, express $\frac{1}{x^4-1}$ in terms of partial fractions.

Solution

You notice x^4-1 is a different of two squares that can be reduced to quadratic factors as show below.

Using,

$$(a^2 - b^2) = (a + b)(a - b)$$

$$(x^4 - 1) = ((x^2)^2 - 1) = (x^2 + 1)(x^2 - 1)$$

Similarly,

$$(x^2 - 1) = (x + 1)(x - 1)$$

Hence,

$$(x^4 - 1) = (x^2 + 1)(x + 1)(x - 1)$$ And the fraction is,

$$\frac{1}{x^4 - 1} = \frac{1}{(x^2 + 1)(x + 1)(x - 1)}$$

This expression is a repeated linear and irreducible quadratic factor and hence, it will decompose in a general form of,

$\frac{g(x)}{(ax^2+bx+c)(dx+ex)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e} + \frac{D}{fx+g}$ applying the general for formula to the equation we get

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\frac{1}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

Multiplying each term by the LCM, $LCM = (x^2 + 1)(x + 1)(x - 1)$

$$\frac{1(x^2+1)(x+1)(x-1)}{(x^2+1)(x+1)(x-1)} = \frac{(Ax+B)(x^2+1)(x+1)(x-1)}{x^2+1} + \frac{C(x^2+1)(x+1)(x-1)}{x+1} + \frac{D(x^2+1)(x+1)(x-1)}{x-1}$$

After cancellation we get

$$1 = (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1)$$

We then solve for A, B , C and D.

By substituting $x = -1$, A, B, and D are eliminated since, $(x+1) = 0$ and A, B and D has a factor of $(\mathrm{x}+1)$. We hence solve for C as follows

$$1 = 0 + (12+1)(-1-1) + 0$$

$$C = \frac{-1}{4}$$

By substituting x=1, A, B, and C are eliminated since, $(\mathrm{x}-1) = 0$ and A , B and C has a factor of $(\mathrm{x}+1)$. we hence solve for D as follows.

$$1 = D(2)(2)$$

$$D = \frac{1}{4}$$

By substituting$x = 0$, A is eliminate since we have a term Ax and hence B can be solved.

$$1 = B(1)(1)(-1) + C(1)(-1) + D(1)(1)$$

$1 = -B + -C + D$, by then substituting , C=-1/4 and D=1/4 we get

$$B = -1 + C + D$$

$$B = -1 + 1/4 + 1/4$$

$$B = \frac{-1}{2}$$

Since A is the only unknown constant substituting x for any other constant can enable us solve for A. Using, $x = 2$ we get,

$$1 = ((2A + B)(3)(1) + C(5)(1) + D(5)(3)$$

$$1 = (4)(2A + B) + 5C + 15D$$

$$1 = 8A + 4B + 5C + 15D \quad But\ B = -1/2, C = -1/4\ and\ D = 1/4$$

$$1 = 8(A) + 4X(-1/2) + (-1/4) + 15(1/4)$$

$$A = 0$$

Therefore; A = 0, B = - ½ , C = ¼ and D = ¼

Substituting back to the main equation,

$$\frac{1}{(x^2+1)\,(x+1)\,(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$ we get

$$\frac{1}{(x^2+1)\,(x+1)\,(x-1)} = \frac{0x+{}^{-1}/_2}{x^2+1} + \frac{{}^{-1}/_4}{x+1} + \frac{{}^{1}/_4}{x-1}$$

$$\frac{1}{(x^2+1)\,(x+1)\,(x-1)} = \frac{{}^{-1}/_2}{x^2+1} + \frac{{}^{-1}/_4}{x+1} + \frac{{}^{1}/_4}{x-1}$$

Or

$$\frac{1}{(x^2+1)\,(x+1)\,(x-1)} = \frac{-1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}$$

Exercise 4.3

Work in pairs.

Express each of the following fractions in partial fractions.

1. $\frac{x^2}{x^2-1}$
2. $\frac{x}{(x^2+2x-3)}$
3. $\frac{x^2-2}{(x^2+2x+1)(x^2-x-2)}$
4. $\frac{x^2-2}{(x^2+2x+1)(x^2-x-2)}$
5. $\frac{x}{(x^2+1)(x^2+2)}$
6. $\frac{2x-2}{(x-1)^2(x^2+x+2)^3}$
7. $\frac{x}{(x^2+1)(x^2+2)}$
8. $\frac{x+3}{(x^2-1)(x^2+5)}$
9. $\frac{2x-2}{(x^2+x+4)(x+2)}$

10. $\frac{x+7}{(x^2-x-6)}$

11. $\frac{8x^3+13x}{(x^2+2)^2}$

12. $\frac{3x^2+4x+4}{4x+x^3}$

13. $\frac{x^2+2x+3}{(x-6)(x^2+4)}$

14. $\frac{8-3x}{(10x^2+13x-3)}$

Application of partial fractions in integration

Partial fractions generate fractions that can be integrated through the method of substitution where the denominator has power of x as x^n and the numerator has a term of x with the highest power being x^{n-1}. These expressions enable cancellation after the first step of substitution during integration. In this substitution u is the denominator.

Example 1

In groups, evaluate $\int \frac{2x-2}{(x^2+x+4)(x+2)} dx$

Solution

You notice the other methods of integration cannot determine the integral of this expression without decomposing it into partial fractions. Hence, Express $\frac{2x-2}{(x^2+x+4)(x+2)}$ as a partial fraction

Check if $(x^2 + x + 4)$ is reducible. If reducible to linear factors then the discriminant $b^2 - 4ac > 0$

In the quadratic equation $(x^2 + x + 4)$, *a=1, b=1* and *c=4.*

Substituting to discriminant we get,

$$1^2 + {}^{-}4\times 1\times 4 > 0$$

1+-16 > 0

-15>0, this contradict the condition for real roots and hence (x^2+x+4) is irreducible.

Using the general decomposition for irreducible quadratic and linear denominator, we apply;

$\frac{g(x)}{(ax^2+bx+c)(dx+ex)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$**linear and irreducible quadratic** denominator

On applying general decomposition

$\frac{g(x)}{(ax^2+bx+c)(dx+ex)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$**linear and irreducible quadratic** denominator

$$\frac{(2x-2)}{(x^2+x+4)(x+2)} = \frac{(Ax+B)}{(x^2+x+4)} + \frac{C}{(x+2)}$$

Multiply each term by the LCM.

$LCM = (x^2+x+4)(x+2)$

$$\frac{(2x-2)(x^2+x+4)(x+2)}{(x^2+x+4)(x+2)} = \frac{(Ax+B)(x^2+x+4)(x+2)}{(x^2+x+4)} + \frac{C(x^2+x+4)(x+2)}{(x+2)}$$

After cancellation we get,

$$(2x-2) = (Ax+B)(x+2) + C(x^2+x+4)$$

On substituting x =-2, A and B are eliminated since they are multiplied by (x+2). Hence,

2× ⁻2 - 2= 0 + C (⁻2² -2 +4)

-6 = 6C

C=-1

On substituting $x = 0$ and $C = -1$, A is eliminated since it is multiplied by x. Hence we can solve for B for will be the only remaining unknown in the expression

$$(2x-2) = (Ax+B)(x+2) + C(x^2+x+4)$$

2×0 =B (2) =C (4)

-2 = 2B +4 C,   but c= -1

-2 = 2B -4

2B = 2

B=1

On substituting $x = 2$, $B = 1$ and $C = -1$, we solve for A since it is the remaining unknown in the expression $(2x-2) = (Ax+B)(x+2) + C(x^2+x+4)$

0= (A+1)(3) +(-1)(6)

0 = 3A – 3

A= 1

Hence, $A = 1$, $B = 1$ and $C = -1$

Substituting, $A = 1$, $B = 1$ and $C = -1$ to the main equation,

$\frac{(2x-2)}{(x^2+x+4)(x+2)} = \frac{(Ax+B)}{(x^2+x+4)} + \frac{C}{(x+2)}$ we get

$$\frac{(2x-2)}{(x^2+x+4)(x+2)} = \frac{(x+1)}{(x^2+x+4)} + \frac{-1}{(x+2)}$$

Or

$$\frac{(2x-2)}{(x^2+x+4)(x+2)} = \frac{(x+1)}{(x^2+x+4)} - \frac{1}{(x+2)}$$ this implies,

$$\int \frac{2x-2}{(x^2+x+4)(x+2)}dx = \int (\frac{(x+1)}{(x^2+x+4)} - \frac{1}{(x+2)})dx$$

$$\int \left(\frac{(x+1)}{(x^2+x+4)}\right)dx + \int (\frac{1}{(x+2)})dx$$

But,

$$\int (\frac{1}{(x+2)})dx = \ln|(x+2)| + c$$

To integrate $\int \left(\frac{(x+1)}{(x^2+x+4)}\right)dx$ we use the general integration formula and u-substitution such that $u = (x + {}^1/_2)$

$\int \left(\frac{(x+1)}{(x^2+x+4)}\right)dx = \int \left(\frac{(x+1)}{\left((x+\frac{1}{2})^2+\frac{15}{4}\right)}\right)dx = \int \left(\frac{\left(u-\frac{1}{2}+1\right)}{\left(u^2+\frac{15}{4}\right)}\right)du$ substituting to the general formula we get

$$\int \left(\frac{\left(u-\frac{1}{2}+1\right)}{\left(u^2+\frac{15}{4}\right)}\right)du = \frac{1}{2}ln\left|\left(u^2+\frac{15}{4}\right)\right| + \frac{1}{\sqrt{15}}\arctan\frac{u}{0.5\sqrt{15}} + c$$

But $u = (x + {}^1/_2)$

Hence,

$$\int \left(\frac{(x+1)}{(x^2+x+4)}\right)dx = \frac{1}{2}ln\left|\left((x+{}^1/_2)^2+\frac{15}{4}\right)\right| + \frac{1}{\sqrt{15}}\arctan\frac{(x+{}^1/_2)}{0.5\sqrt{15}} + c$$

$$\int \left(\frac{(x+1)}{(x^2+x+4)}\right)dx + \int (\frac{1}{(x+2)})dx$$
$$= \ln|(x+2)| + \frac{1}{2}ln\left|\left((x+{}^1/_2)^2+\frac{15}{4}\right)\right|$$
$$+ \frac{1}{\sqrt{15}}\arctan\frac{(x+{}^1/_2)}{0.5\sqrt{15}} + c$$

Expressing partial fractions as single fractions

Partial fractions can be expressed as a single fraction in the method similar to the addition of subtraction of other fractions

Example

In pairs, express $x + \frac{x}{x^2-1}$ as a single fraction.

Solution

Determine the LCM of the denominators

$LCM = 1(x^2 - 1)$

Divide each denominator by LCM and multiply by numerator and simplify,

$$x + \frac{x}{x^2 - 1}$$

$= \frac{x(x^2-1)+x}{x^2-1}$

$$= \frac{x(x^2 - 1) + x}{x^2 - 1}$$

$= \frac{x^3-x+x}{x^2-1}$

$= \frac{x^3}{x^2-1}$

Exercise 4.4

Work in pairs.

Find the values of integral in question 1-7.

1. $\int \frac{8}{(3x^2+7x^2+4x)} dx$

2. $\int \frac{3x^2+1}{(x+1)(x-5)^2} dx$

3. $\int \frac{3x^2-3x}{(x-2)(x+4)} dx$

4. $\int \frac{x^4+2}{x^3+9x} dx$

5. $\int \frac{8-3x}{10x^2+13x-3} dx$

6. $\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)} dx$

7. $\int \sec x \, dx$

8. Express the following fractions as a single fraction
   a. $\frac{1}{(x-1)^2} + \frac{4}{(x-1)^3}$
   b. $1 + \frac{4}{x^4+1}$

UNIT5

VECTORS

Introduction

The concept of vector is a familiar subject. However, we need to review some of the concepts and introduce vectors in **i, j** and **k** terms in vectors. A v**ector** is a physical quantity with both **magnitude** and **direction**. Quantity that have only magnitude no direction are called scalar quantities.

Activity 1

1. List at least three examples of vector quantities.
2. List least three examples of scalar quantities.
3. List three vector quantities obtained by defining direction of scalar quantities.

Examples of **scalar quantities** includes distance, mass, speed, temperature, pressure, and energy and among other things. When direction of a scalar quantity is given is specified the quantity becomes a vector quantity such quantities include,

Scalar quantity	Vector quantity obtained by specifying direction
Distance Speed Weight	Displacement Velocity Weight

Examples of vector quantities are hence displacement, velocity, weight, acceleration force among others. A vector is illustrated by a directed line that specify the **initial point [tail]** and **terminal point [head]** of a vector

Figure 5.1 below shows a vector **AB** on a Cartesian plane

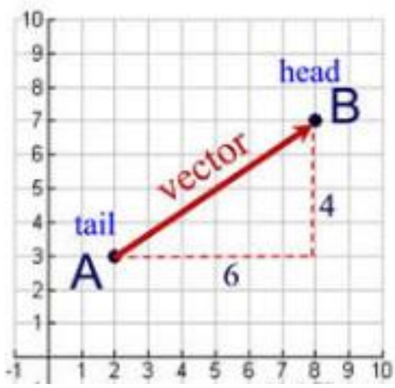


Figure 5.1

When hand written vector written by capital letter with a line above it $\overrightarrow{AB}$ when typed it is in bold **AB** where A is initial point [tail]. B is terminal point [head] Or a small letter a with a line below or in bold **a** . In short form a vector can be represented in form of;

i. column vectors
ii. **i**, **j** and **k** notation

The vector above can be expressed as a column vector as

$$\mathbf{AB} = \begin{pmatrix} \text{Change in x component} \\ \text{change in y component} \end{pmatrix}$$

Hence

$$\mathbf{AB} = \begin{pmatrix} 8-2 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

i, j and k notation

i, j and **k** are unit vectors where **i** is one unit on x- axis **j** is one unit on y-axis and **j** is one unit on z-axis. Figure 5.2 below shows the unit vectors **i, j** and **k** in a 3 dimension Cartesian plane.

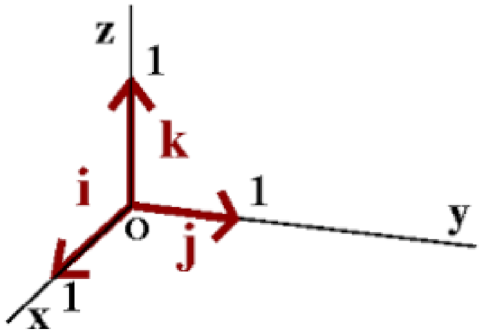


Figure 5.2

Vector **AB** in the figure above represents 5 units of x axis hence equivalent to 5**i,** 4 units on y axis hence equivalent to 4**j. O** units on z axis hence equivalent to 0**k**.

Example 1

Express vector $\boldsymbol{AB}$ in figure 5.1 above in terms of **i, j** and **k.**

Solution

AB can be written as $\boldsymbol{AB} = 5\boldsymbol{i} + 4\boldsymbol{j} + 0\boldsymbol{k}$

$$\boldsymbol{AB} = 5\boldsymbol{i} + 4\boldsymbol{j}$$

Position vector in 3 dimensions

Position vector is a vector of a given point on a Cartesian plane from the origin (0, 0) or (0, 0, 0), position vector of a general point P(x, y, z)is written as **OP** or **p** where **O** is the origin.

Figure 5.3 below shows the position vector of point **P in three dimensions,**

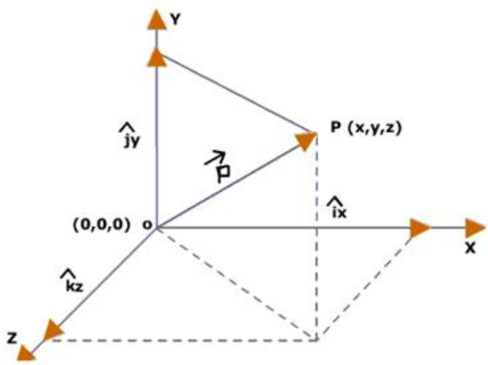


Figure 5.3

The position vector **OP** can be written in different ways, this include

i. In **column vector OP** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

ii. **In unit vector** $\boldsymbol{OP} = \boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

Equal vectors

Two vectors are said to be equal if they have **equal magnitude and direction. In** figure 5.4 below vector **AB** =vector **CD** since they face to the same direction and are of equal magnitude.

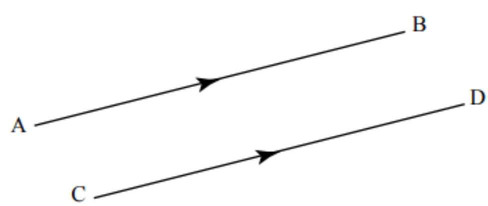


Figure 5.4

If direction of a vector **AB** = **a** is changed, the vector sign changes. Figure 5.5a below show vector **AB** =**a** and vector **BA** =**-a.** You notice the order of points **A** and **B.** Changes in the vector to form vector **BA.**

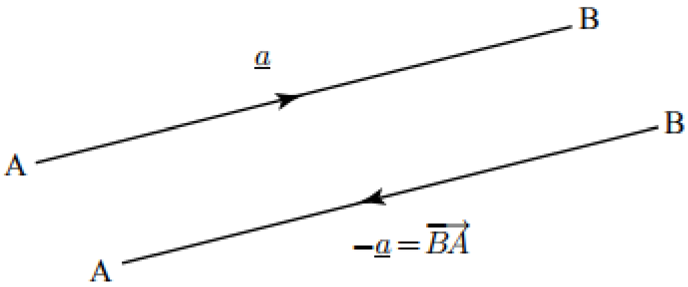


Figure 5.5

Example 2

In pairs, study the figure below 5.6 and answer the questions that follow.

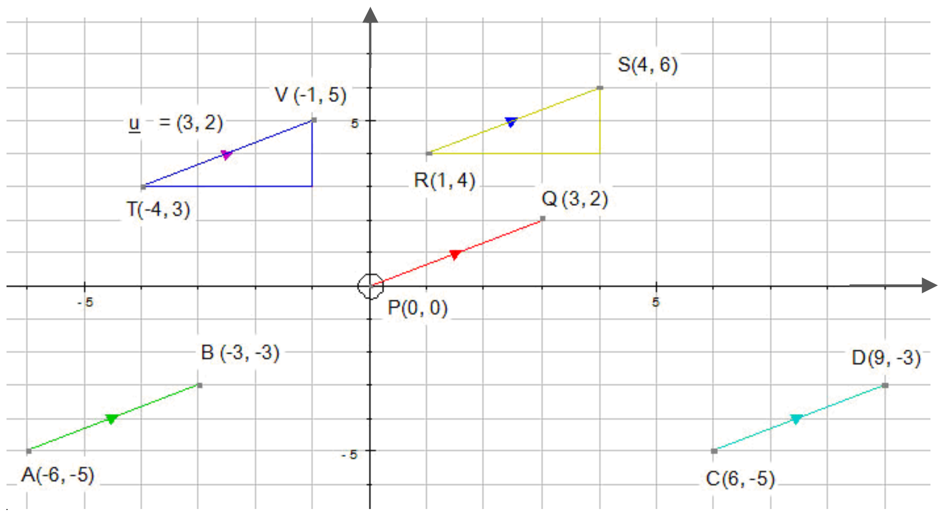


Figure 5.6

a. Determine the column vectors of vectors **AB, CD, PQ, TV and RS** in the figure.
b. Determine the vectors of vectors in **i, j** and **k** form **AB, CD, PQ, TV and RS** in the figure.
c. What do you notice about these vectors?
d. Why is vector referred as space free?

Solution.

They can be represented in column vectors as

AB = CD= PQ= TV= RS= $\binom{-4-1}{3-5} = \binom{3}{2}$

Or in unit vector as **u=a= b=c=d**=3**i**+2**k**

In the figure 5.5b below all the vectors indicated have same magnitude and direction and hence they are equal, because they represent same unit on x axis and same units on y axis.

It is important to note that the **a** vector is defined by its direction and magnitude on space and not on its location in space. A vector is hence said to be a **space free vector**.

Additional vectors

Vectors are added in a particular way known as the triangle law. Consider a scenario where an automated convey belt is carrying material from store at point **A** to a worker at appoint **C**. The belt is made in such a way that it can operate in route **AC** and if route **AC** fails an alternative route through point **B** can be followed from **A** to B then from **B** to **C**. Since the distance between point **A, B** and **C** have a magnitude and the movement **is** specified (is in specific direction) we represent vectors **AB, BC**, and **AC** as shown in figure 5.7 below.

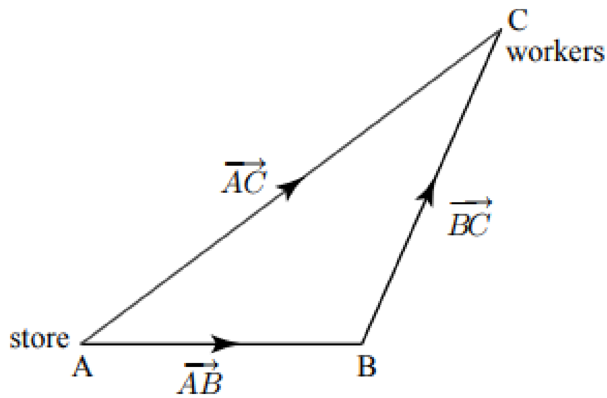


Figure 5.7

Since travelling from **A** to **B** and to **C** is the same as travelling directly from **A** to **C** then the **vectors represented are equal and AC=AB+BC.**

on adding two different vectors at different space as shown in figure 5.8 below you move one vector **a** to join other vector **b** such that the head of **a** is the tail of **b** and a resultant vector **c** = **a** + **b** is clearly illustrated as shown in figure 5.9 that follow below.

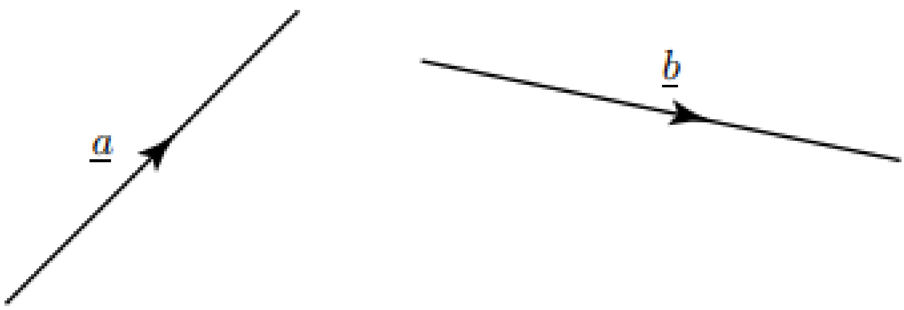


Figure 5.8 before adding

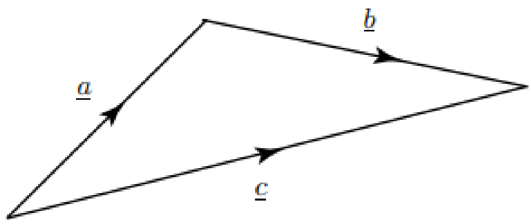


Figure 5.9after adding

Addition of vectors in Cartesian plane

The aspect of moving a vector from one space to another space can also be used to add vectors in the Cartesian plane. This is done by retaining magnitude and direction and ensuring the head of one vector becomes the tail of the other vector.

Example 3

In pairs, study figure 5.10 below and answer the questions that follow.

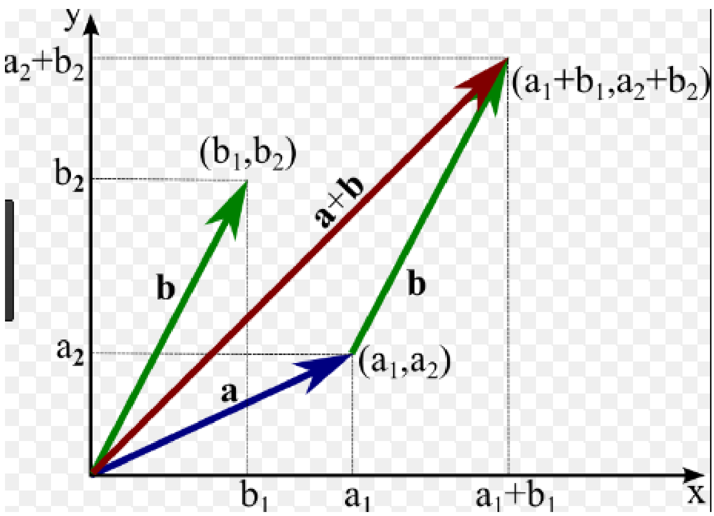


Figure 5.10

a. Write vector c as a column vector in terms of a_1, b_1, a_2 **and** b_2**.**
b. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ find the value of $\mathbf{a} + \mathbf{b}$

Solution

To get **a** + **b** move **b** such that its tail contact the head of **a** .You notice if

$$\mathbf{a} = \mathbf{OA} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}$$

$$\mathbf{b} = \mathbf{OB} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$$

Then, $\mathbf{a} + \mathbf{b} = \mathbf{OA} + \mathbf{OB} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 \\ \mathbf{a}_2 + \mathbf{b}_2 \end{pmatrix}$

In unit vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

$\mathbf{b} = b_1 i + b_2\mathbf{j}$, then

$\mathbf{a} + \mathbf{b} = a_1\mathbf{i} + b_1\mathbf{i} + a_2\mathbf{j} + b_2\mathbf{j}$

$\mathbf{a} + \mathbf{b} = (a_1+b_1)\mathbf{i} + (a_2+b_2)\mathbf{j}$

In 3 dimensions

If; $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$\mathbf{a}+\mathbf{b} = (a_1+b_1)\mathbf{I} + (a_2+b_2)\mathbf{j} + (a_3+b_3)\mathbf{k}$

Example 4

Given $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ find in pair,

a. **2a**
b. $\boldsymbol{a} + \boldsymbol{b}$

Solution

a. $2\boldsymbol{a} = 2(2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) = \boldsymbol{4i} + 6\boldsymbol{j} + 6\boldsymbol{k}$
b. $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{3i} + 3\boldsymbol{k} + 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} = 4\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}$

Exercise 5.1

Work in groups of four students.

1. Given that **a** = 6**i** - 8**j**, **b** = -8**i**+3**j** and **c** = 26**i** + 19**j**. Find the value of m and n If m**a** + n**b** + **c** = 0
2. Given that **a-b** = 2**c** , **a+b** = **c** and **c** = 3**i**+4**j** determine the value of **a** and **b.**
3. Given that **a**=2**i** + 3**k**, **b** = **j+k** find the value **of ;**
   a. a + b
   b. 2b
   c. b + a
   d. 4a
4. Given that **a**= $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$and **b**=$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and **c**=$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$
   Find the following vectors in form of **i, j** and **k**
   a. 2a
   b. 2a+b
   c. **a+b+c**

Magnitude of vector (Length of a line and vector)

Magnitude of a vector **a** refers to the length of the vector line its **symbol** is $|\boldsymbol{a}|$. if the position of a line in a plane is known the length of the line can be obtained using Pythagoras theorem. The position vector of such points can also be identified and hence the vector for the two points.

Example 5

In groups of three, determine the length of the line **OP** in the figure 5.11 below given the coordinates of P(2,3,5)

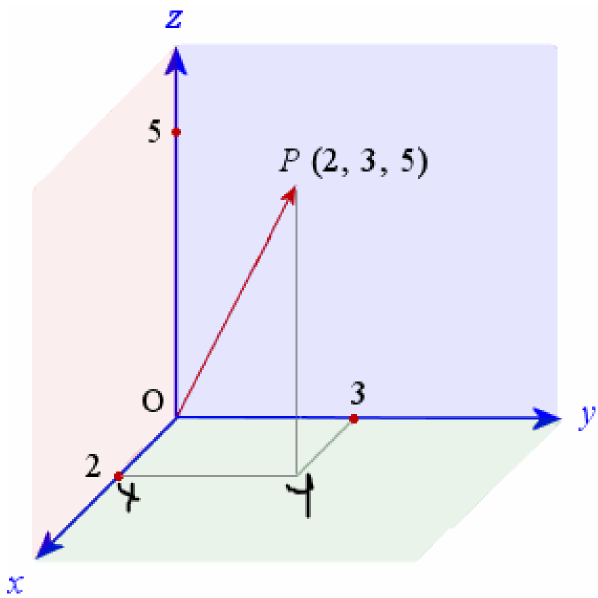


Figure 5.11

Solution

Using right triangle OXY and Pythagoras theorem

$\mathbf{OY}^2=2^2+3^2$

Using right triangle OYP and Pythagoras theorem

$|\boldsymbol{OP}|^2=|OY|^2+|YP|^2$

$|\boldsymbol{OP}|^2=2^2+3^2+5^2$

$|\boldsymbol{OP}|=\sqrt{2^2+3^2+5^2}=\sqrt{38}$

You notice that for P(2,3,5), **OP**=$\begin{pmatrix}2\\3\\5\end{pmatrix}$ and $|OP|$=$\sqrt{2^2+3^2+5^2}$

In general

for a vector **a**=$\begin{pmatrix}x\\y\\z\end{pmatrix}$, its length/magnitude $|a|$ =$\sqrt{(x^2+y^2+z^2\,)}$

Example 6

Given the point Q(2,3,4) and R(0,4,7) are points on a graph. In groups of three find

i. vector **PQ** in **i, j** and **k**
ii. The magnitude $|\boldsymbol{PQ}|$

Solution

PQ = OQ - OP

PQ =$\begin{pmatrix}0\\4\\7\end{pmatrix}$-$\begin{pmatrix}2\\3\\4\end{pmatrix}$=$\begin{pmatrix}-2\\1\\3\end{pmatrix}$

$|\boldsymbol{PQ}| = |\boldsymbol{p}| = |-2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}|$

Using the general case , magnitude of **a**=$\begin{pmatrix}x\\y\\z\end{pmatrix}$ is $|a|$ =$\sqrt{(x^2+y^2+z^2\,)}$

|**PQ**|= $\left|\begin{pmatrix}-2\\1\\3\end{pmatrix}\right| = \sqrt{(-1^2+1^2+z^2\,)} = \sqrt{14}$

Scalar multiple of a vector

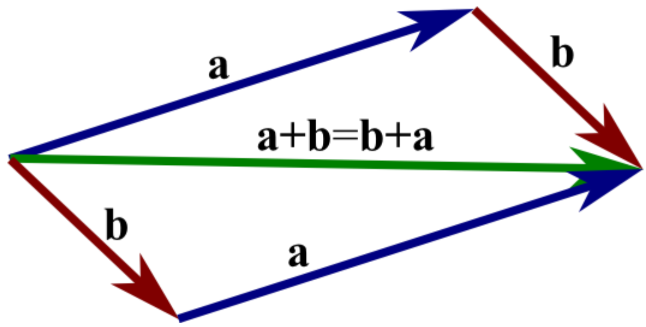
If two vectors are a scalar multiple of each other, then the vector are parallel.

Distributive law of vectors

It states that $K(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) = k\boldsymbol{a}+k\boldsymbol{b}+k\boldsymbol{c}$

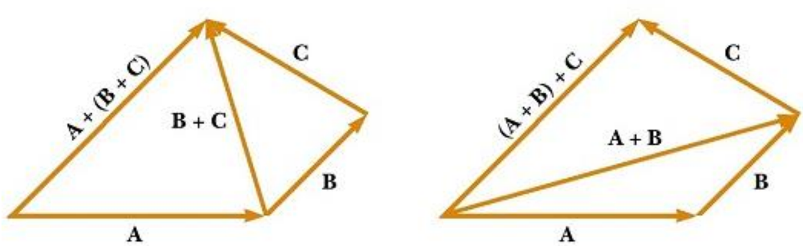
Commutative law of vector

States that $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}$



Associative law of vector

States that $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$



The unit vector

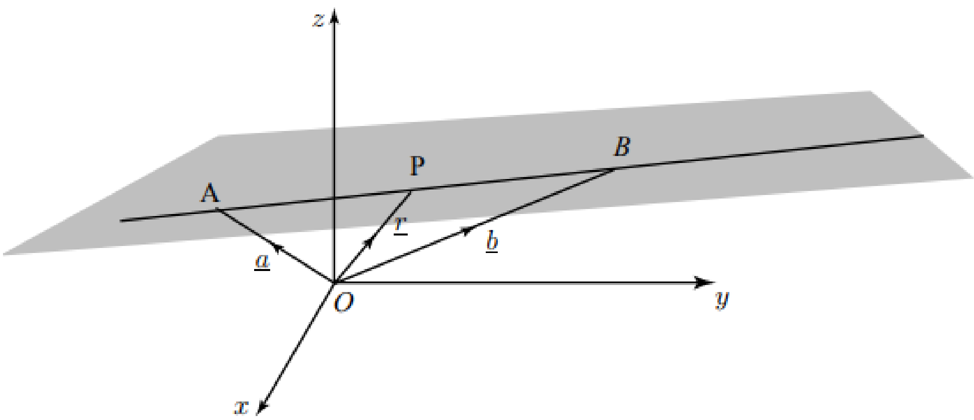
Unit vector of a vector **a** is the vector **a** divided by its magnitude |**a**|

Exercise 5.2

1. Given that point **A** and **B** has coordinate(2,3,7) and **B**(4,0,2) find
   i. **AB** in terms of **i, j** and **k**
   ii. **BA** in terms of **i, j** and **k**
   iii. |**OB**|
   iv. |**AB**|
   v. **Unit vector of AB**
2. Given that **a** =2**i**+3**j**+4**k** find the unit vector of a
3. Given that **a** =2**i**+3**j**+4**k** , **q** =2**i**+**k** and **r** = 2**k** . find the value of the following vectors in terms of **i, j** and **k**
   i. 2**r**
   ii. 2(**p**+**q**)+**r**
   iii. **p**-2 (**q**+**r**)

The vector equation of line

Consider ***Figure 5.12*** below.



In the figure the line ABP is a straight line in three dimensional space and points A and B have known coordinates $A(x_1,y_1,z_1)$ and B

(x_2,y_2,z_2). Their position vectors are **a** and **b** respectively. Point P is an arbitrary general point P(x, y, z) and its position vectors is **r.**

Vector **AB**= **r** + **a**

Vector **AP** = k (**a + b)**

AP=k(**b+a**)

OP = p= **a**+k(**b +-a**)

The expression **p** =**a +** k(**b**-**a**) is called **vector equation** of a line through point A and B with position vectors **a,** and **b** respectively.

Example 7

In groups, write vector equation of a line passing through the point whose position are $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 7\boldsymbol{i} + 5\boldsymbol{j}$.

Solution

Consider **general case**. The expression $\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$ is called **vector equation** of a line through point A and B with position vectors **a** and **b** respectively.

$$\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$$

$$\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} + k[7\boldsymbol{i} - 5\boldsymbol{j}] - [3\boldsymbol{i} = 2\boldsymbol{j}]$$

$$\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} + k([7\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{j} - 2\boldsymbol{j}]$$

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + k\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Example 8

In groups of three, write vector equation of the line passing through the points with position vector $\boldsymbol{a} = 5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{i} - 2\boldsymbol{k}$

Solution

$$\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$$

$$\boldsymbol{p} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{p} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} -3 \\ 5 \\ -5 \end{pmatrix}$$

Exercise 5.3

Work in pairs.

1. Write the vector equation of the line which passes through the position vector below.
   i. $\boldsymbol{a} = 5\boldsymbol{j} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $2\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$
   ii. $\boldsymbol{p} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ *and* $\boldsymbol{r} = 2\boldsymbol{k}$
   iii. **a**=2**i** and b=2**k**=**j**

2. Find the vector equation of a line that pass through the following points
   i. A(2,3,4) and B(1,1,1)
   ii. A(2,3,0) and B(0,0,0)
   iii. A(5,6,7) and B(2,2,1)
   iv. A(7,8,9) and B(-1,7,0)

Cartesian form of the equation of straight line

Consider a line passing through general point $P(x, y, z)$ and given points A(x_1, y_1, z_1)and B(x_2, y_2, z_2). The **vector equation** of this line is

$$\boldsymbol{p} = \boldsymbol{a} + k(\boldsymbol{b} + \boldsymbol{a})$$

Substituting this equations with column vectors for vector equation we obtain

$$\boldsymbol{P}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+k\left(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}-\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\right)$$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x1\\y1\\z1\end{pmatrix}+\mathrm{k}\begin{pmatrix}x2-x1\\y2-y1\\z2-z1\end{pmatrix}$

Separating the **i, j** and **k components** of vector we get three equations called the parametric equations of a line. The **general parametric equations** of a line passing through points A($x_1,\ y_1, z_1$) and B($x_2,\ y_2, z_2$) are;

$$x=x_1+k(x_2-x_1)$$

$$y=y_1+k(y_2-y_2)$$

$$z=z_1+k(z_2-z_1)$$

Making k the subject of parametric equations we get expression for Cartesian equation.

$$\mathrm{K}=\frac{\mathrm{x-x1}}{\mathrm{x2-x1}}=\frac{\mathrm{y-y1}}{\mathrm{y2-y1}}=\frac{\mathrm{z-z1}}{\mathrm{z2-z1}}$$

This equation is called the Cartesian equation of a straight line passing through points A(x_1,y_1,z_1)and B(x_2,y_2,z_2) is $\frac{\mathrm{x-x1}}{\mathrm{x2-x1}}=\frac{\mathrm{y-y1}}{\mathrm{y2-y1}}=\frac{\mathrm{z-z1}}{\mathrm{z2-z1}}$

Example 9

In groups,

i. Write the Cartesian equation of a straight line that pass through points A(5,6,7) and B(1,2,2)

ii. State the equivalent vector equation.

Solution

Consider general Cartesian equation of a straight line passing through points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$$(5, 6, 7) = (x_1, y_1, z_1)$$

$$(1, 2, 2) = (x_2, y_2, z_2)$$

The Cartesian equation is hence;

$$\frac{x-5}{1-5} = \frac{y-6}{1-6} = \frac{z-7}{2-7}$$

$$\frac{x-5}{-4} = \frac{y-6}{-5} = \frac{z-7}{-5}$$

To obtain the vector equation $\boldsymbol{p} = \boldsymbol{a} + k(\boldsymbol{b} + \boldsymbol{a})$ consider the general column vector equation

$$\mathbf{P}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}+\mathrm{k}\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right)$$ Substituting to this equation we get

$$\mathbf{P}=\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}+\mathrm{k}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}\right)$$

$$\mathbf{P}=\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}+\mathrm{k}\begin{pmatrix} -4 \\ -5 \\ 5 \end{pmatrix}$$

Exercise 5.4

Work in pairs.

1. Answer these questions.
   a. Write down the Cartesian equation for a line passing through point P (9,3,2) and Q(4,5,-1)
   b. Write the vector equation of the line in a above

2. Find the parametric equation of the line which passes through the point A(-2,1,0) and B([1,3,5) on a 3 dimension Cartesian plane
3. Find the Cartesian equation of a line that pass through point A(1,-2,4) and is parallel to position vector $2\boldsymbol{i} + \mathbf{4}j - 4\boldsymbol{k}$
4. Given that $\boldsymbol{a} = -2\boldsymbol{i} + +4\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{b} = -1\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k}$
   a. find vector **AB**
   b. find the Vector equation of the line

The scalar product

Scalar product is a method of combining two vectors. As the name suggests the result of a scalar product is a scalar rather than a vector. When two vectors are drawn such that their tail meet at a point and form an angle **θ** as shown in figure 5.13 below, then the scalar product **a** dot **b**, written as $(\boldsymbol{a}.\boldsymbol{b})$ is equals to magnitude of a times magnitude of $\boldsymbol{b}$ times **cosine** of angle **θ.** A scalar product can be considered a scaled projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.

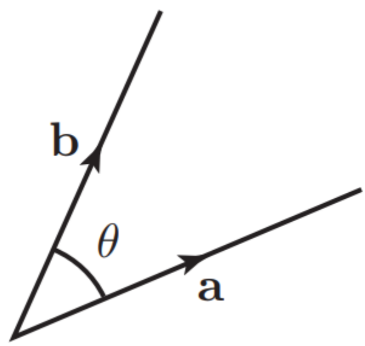


Figure 5.13

The scalar product of **a** and **b** is

a.b = $|\boldsymbol{a}||\boldsymbol{b}|$**cos θ**

Where **a** and **b** are vectors, $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are magnitude/modulus of vector **a** and **b** respective and **θ** Is angle between vector **a** and **b**

Example 10

Calculate the angle between vector **a** whose magnitude$|\boldsymbol{a}|$ = 4 and vector whose magnitude $|\mathbf{b}|$=5 units and they meet at angle 60^0.

a. Calculate the scalar dot product
   i. **a.b**
   ii. **b.a**

b. what do you notice about **a.b** and **b.a** ?

Solution

ai.

$$\mathbf{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\mathbf{a.b} = 4\text{x}5\cos 60^0$$

$$=10$$

aii.

$$\mathbf{b.a} = |\boldsymbol{b}||\boldsymbol{a}|\cos\theta$$

$$\mathbf{b.a} = 5\text{x}4\cos 60^0$$

$$\mathbf{=10}$$

b. Hence **a.b = b.a = 10**
   Scalar product is commutative and hence **a.b = b.a** this mean the order of writing Scalar product vector is not important. Whichever order is used the angle is the same.

Properties of a scalar (dot product)

Commutativity

Scalar product is commutative and hence **a.b=b.a** this mean the order of writing Scalar product vector is not important. Whichever order is used the angle is the same.

Distributive

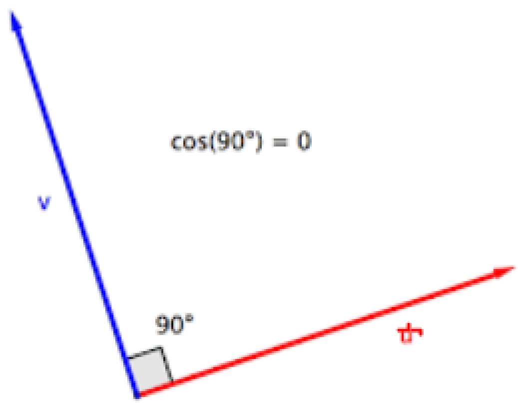
Scalar/dot product is distributive over product this mean it is possible to make the expansion of the form;

a.(b+c)=a.b+a.c

(b+c). a=b.a+c.a

Perpendicular /orthogonal vectors

Consider the perpendicular **a** and **b**



Since $\cos 90° = 0$

a.b=0

In general the dot/**scalar product for orthogonal/perpendicular vectors is 0.**

Example 11

Study figure 5.14 in groups and answer the questions that follow.

i. Determine the value of the scalar product of unit vectors **i.j, i.i, i.k, k.j, j.j, k.j**

ii. Hence determine the value of **a.b** given that **a** $= x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ *and*

b $= x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

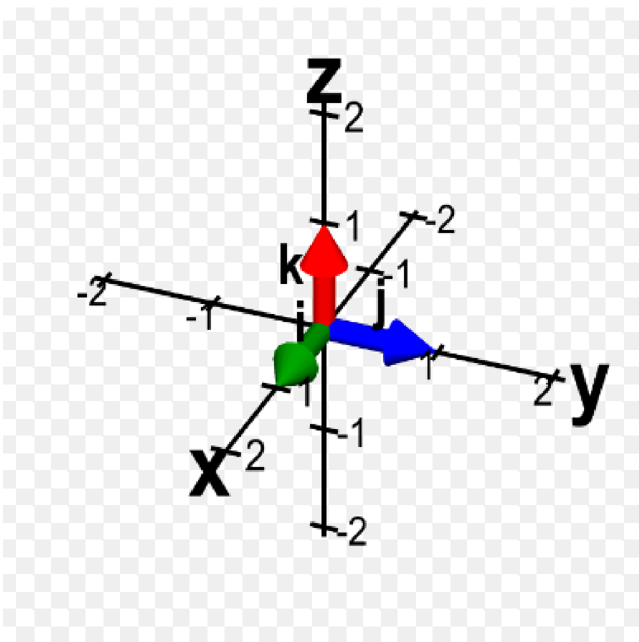


Figure 5.14

Solution

Since these unit vectors, their magnitude is 1. Therefore, $|\boldsymbol{i}| = |\boldsymbol{j}| = |\boldsymbol{k}| = 1$. Working out the scalar products of different unit vectors we get;

a.b = $|\boldsymbol{a}||\boldsymbol{b}|$**cos θ**

i.i=1×1 cos $90^0=1$

j.j=1×1 cos $90^0=1$

k.k1.1 cos $90^0=1$

i.j=1×1 cos$90^0=0$

i.k =1×1 cos $90^0=0$

j.k=1×1 cos$90^0=0$

Applying this scenario to position vectors of general points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) we get

a =x_1**i**+y_1**j**+z_1**k**

b =x_2**i**+y_2 **j**+z_2**k**

a.b = (x_1i+y_1j+z_1k) .(x_2i+y_2j+z_2k)on expanding it we get

a.b = x_1x_2**i.i**+x_1y_2**ij**+x_1z_1**ik**+x_2y**ij**+y_1y_2**ij** + y_1y_2**jk**+x_2z_2**kj**+z_1z_2**kk**

Since, **i.i**=0, **k.k**=0 and **j.j**=0 the expansion simplify to.

a.b $= x_1x_2 + y_1y_2 + z_1z_2$

In general if,

a =x_1**i**+y_1**j**+z_1**k** and **b** =x_2**i**+y_2 **j**+z_2**k** the **a**= $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and **b** = $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

and their scalar product is;

a.b= $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$

The scalar product is hence sum of product of coefficient of **i** and **i** ,**j** and **j**, and **k** and **k**

Example 12

In pairs, find the value of **a.b** given that **a** = 2**i** + 3**j**+ **4**k and **b** = 4**i** + 2**k**.

Solution

$$\textbf{a.b}=\begin{pmatrix}2\\3\\4\end{pmatrix}.\begin{pmatrix}4\\0\\2\end{pmatrix}=2\times4+3\times0+4\times2$$

$$=8+0+8\ =\ 16$$

Exercise 5.5

Work in pairs.

1. Find the scalar product of vector **a** and **b** given $\boldsymbol{a}=2\boldsymbol{i}+5\boldsymbol{j}\ +$ $4\boldsymbol{k}$ and $\boldsymbol{b}=4\boldsymbol{i}+3\boldsymbol{j}+7\boldsymbol{k}$
2. Points a and b have the coordinates A(-6,3,-11) and B(12,0,4). Find the value of scalar product of their positions vector **OA.OB**
3. Given that $\boldsymbol{a}=4\boldsymbol{i}+9\boldsymbol{j}$ and $\boldsymbol{b}=3\boldsymbol{i}+2\boldsymbol{j}$, find the scalar products
    - **a. a.b**
    - **b. b.a**
    - **c. a.a**
    - **d. b.b**
4. Simplify 5**i**.8**j**
5. Given 3 dimensional vectors $\boldsymbol{a}=4\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{b}=2\boldsymbol{i}-\boldsymbol{j}+$ $11\boldsymbol{k}$ find the scalar products
    - **i. a.b**
    - **ii. b.a**
    - **iii. a.a**
    - **iv. b.b**
6. Given that $\boldsymbol{a}\ =3\boldsymbol{i}+2\boldsymbol{j}+8\boldsymbol{k}$ show that **a.a**=r^2
7. Given that column vectors **a**=$\begin{pmatrix}-2\\14\\1\end{pmatrix}$ and **b**=$\begin{pmatrix}2\\-3\\-4\end{pmatrix}$find the value of ;

i. **a.b**
ii. **b.a**
iii. **b.b**
iv. **a.a**

8. The coordinates of three points in a 3 - dimension plane are A(3,2,1) , B(5,4,2) and C(-4,2,1). Find the scalar product **AB.AC**

Application of scalar products

Scalar product is used to investigate whether two vectors are perpendicular/orthogonal, to calculate the angle between two vectors and finding component of one vector in term of another. These used are described in the next part.

Perpendicular vectors

If a and b are perpendicular **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 0$ this is the case because $\theta = 90^0$ and $\cos 90^0 = 0$

a.b=0

Example 13

In pairs, determine whether vectors **a**= 2**i**+4**j** and **b**=-**I** + $\frac{1}{2}$ **j** are perpendicular or not

Solution

If perpendicular **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 0$

$$\mathbf{a.b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} . \begin{pmatrix} -1 \\ 0.5 \end{pmatrix} = 2\text{-}2 = 0$$

Hence, they are perpendicular

Finding the angle between two vectors

Recall, if two vector a and b meet at **θ** then;

1) **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and
2) **a.b**= $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$

if the vectors of **a** and **b** are defined the solution in first equation is equated to the first equation for the solution of the angle.

Example 14

In pairs, find the angle between vectors **a** and **b** if $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$and $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$

Solution

Use the relationships

1) **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and
2) **a.b**= $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$

Substituting in 2

$$\mathbf{a.b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 2\times3+0\times3+4\times4 = 16$$

$$\mathbf{a.b} = 22$$

Substituting in 2

a.b = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

$$|\boldsymbol{a}| = \left|\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right| = \sqrt{(2^2+3^2+4^2)} = \sqrt{29}$$

$|\boldsymbol{b}| = \left|\begin{pmatrix}3\\0\\4\end{pmatrix}\right| = \sqrt{(3^2+0^2+4^2)} = 5$

Substituting **a.b=16,** $|\boldsymbol{a}| = \sqrt{29}$ and, $|\boldsymbol{b}| = 5$ in **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ we get

$22 = 5\sqrt{29}\cos\theta$

$\cos\theta = 0.8171$

$\theta = \cos\theta^{-1}\ 0.8171 = 35.21^0$

Finding component of a vector in term of another

Consider figure 5.15 on the following page.

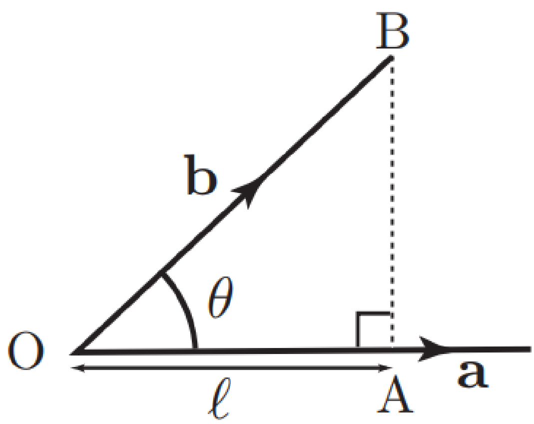


Figure 5.15

In the figure vector b make an angle **θ** to its projection **OA** if l is part of projection of vector **OA** such that **a = OA** and **OB and AB** are perpendicular.

Using **OA** = $\boldsymbol{\ell}$ and trigonometric ratio $\cos\boldsymbol{\theta} = \frac{\boldsymbol{\ell}}{|\boldsymbol{b}|}$

But **a.b** = $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ hence

$\cos\theta = \frac{\mathbf{a.b}}{||\boldsymbol{a}||\boldsymbol{b}||}$

$.. \ \mathbf{l} = \frac{|\boldsymbol{b}|\mathbf{a.b}}{||\boldsymbol{a}||\boldsymbol{b}||} = \frac{\mathbf{a.b}}{||\boldsymbol{a}||} = \mathbf{b.}\ \frac{\mathbf{a.}}{||\boldsymbol{a}||}$

Where $\frac{\mathbf{a.}}{||\boldsymbol{a}||}$is the unit vector of vector **a**. In general the projected vector of **b** onto is vector **b** dot unit vector **a**

$\mathbf{l} = \mathbf{b.}\ \frac{\mathbf{a.}}{||\boldsymbol{a}||}$

Example 15

In pairs, find the component of vector **b** in the direction of **a** if $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{i} + 3\boldsymbol{k}$

Solution

Use component vector $\mathbf{l} = \mathbf{b.}\ \frac{\mathbf{a.}}{||\boldsymbol{a}||}$

$|\boldsymbol{a}| = \left|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right| = \sqrt{(2^2+3^2+1^2)} = \sqrt{14}$

Unit vector **ofa** $= \frac{\mathbf{a.}}{||\boldsymbol{a}||} = \frac{2i+3j+k}{\sqrt{14}} = \begin{pmatrix} 2/\sqrt{14} \\ 3/\sqrt{14} \\ 1/\sqrt{14} \end{pmatrix}$

$\mathrm{L} = \mathbf{b.a} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} = \frac{8}{\sqrt{14}} + \frac{6}{\sqrt{14}} = \frac{3}{\sqrt{14}} = \frac{17}{\sqrt{14}}$

Exercise 5.6

Work in pairs.

1. Investigate whether the following vectors are orthogonal or not $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - 1\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

2. Calculate the angle between vectors **a** and **b** if $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$.

3. Find the component of vectors $\boldsymbol{l} = 3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ in the direction of $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

4. Find the value of $\boldsymbol{a}.\boldsymbol{i}$ where $\boldsymbol{a} = 4\boldsymbol{i} + 8\boldsymbol{j}$ and hence find the angle that **a** make with line y=0.

5. Find the component of vector **a** in direction of **b** if $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 7\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$.

6. Find out if vector $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$ are perpendicular or not.

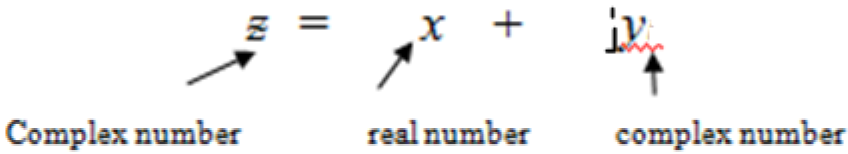
7. Point P(3,2,1) Q(5,4,2) and R(-4,2,1) are on a 3 dimension plane. Find the angle between lines **PQ** and **PR**.

UNIT 6

COMPLEX NUMBERS

Introduction

The concept of complex numbers was introduced in additional mathematics secondary 3 mathematics. Operations on complex numbers have been discussed also. A complex number (z) is a number that has a real number term and an imaginary number $\boldsymbol{i}$, for instance



In this unit we will discuss graphically representation of complex numbers, De Moivre's theorem among other concepts.

Graphical Representation of complex numbers

Graphs of complex numbers are called the **Argand Diagrams**. The Argand diagram is a representations of the **complex plane**. A complex plane is similar to the Cartesian plane. It has;-

a) x-axis:- that represent the real values of complex number.
b) y-axis:- that represent the imaginary number (this is the "y"-axis in Cartesian plane.

When graphing the complex number$z = x + yi$we locate$z$ at a point (x, y)

Figure 6.1 below shows the complex plane with a general complex numbers $z = x + yi$.

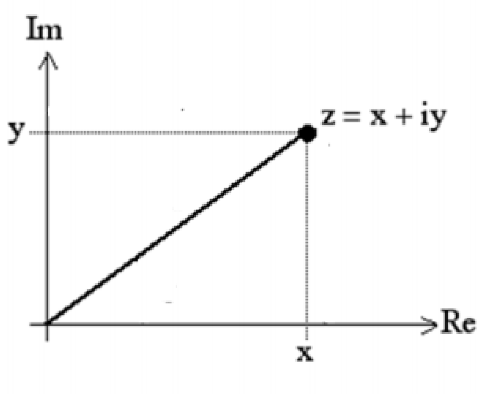


Figure 6.1

Example 1

Draw the graph of the complex number $z = 6 + 4i$

Solution

Plot (6, 4) as (x, Im).

Figure 6.2 below shows $z = 6 + 4i$

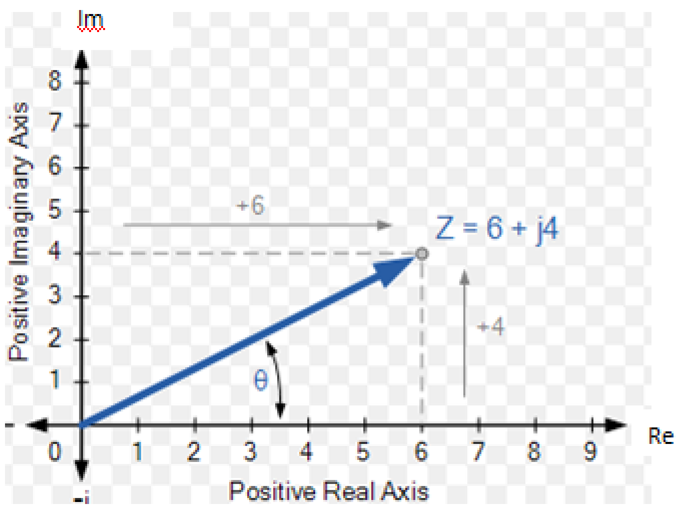


Figure 6.2

Example 2

Draw a Cartesian place and show the position of the following complex numbers.

i) $z_1 = 4+3i$
ii) $z_1 = 0+4i$
iii) $z_3 = -2+3i$
iv) $z_4 = -4$
v) $z_5 = 3-2i$

Solution

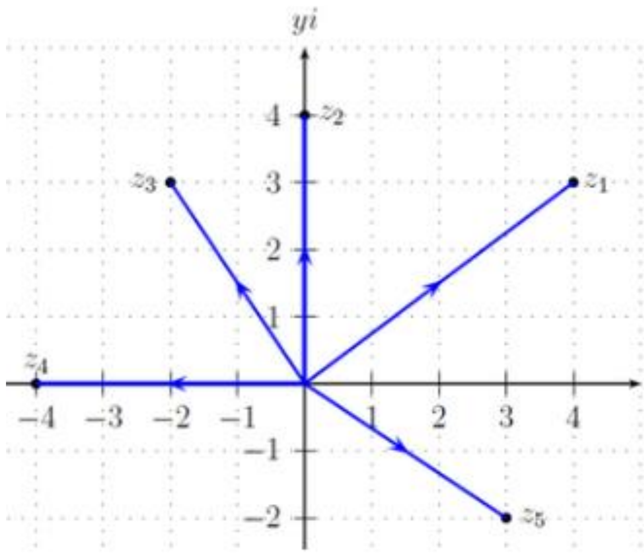


Figure 6.3

Exercise 6.1

Work in groups.

1. Figure 6.4 below shows position of complex numbers z_1, z_2, z_3, z_4 and z_5 starting from ositive real x-axis anticlockwise. Study it and answer the questions that follow.

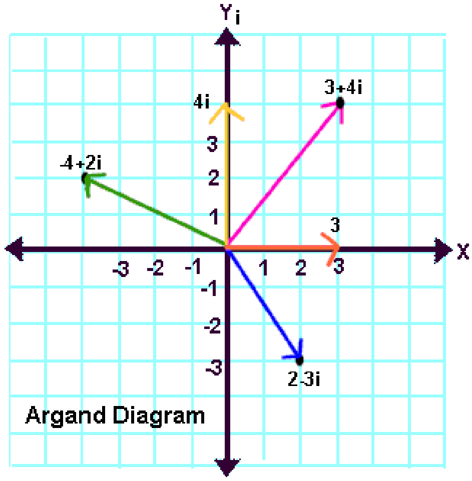


Figure 6. 4

Determine the complex numbers $z_1 \rightarrow z_5$

2. Study figure 6.5 below and write the values of the complex numbers z_1, z_2, z_3, z_4 and z_5

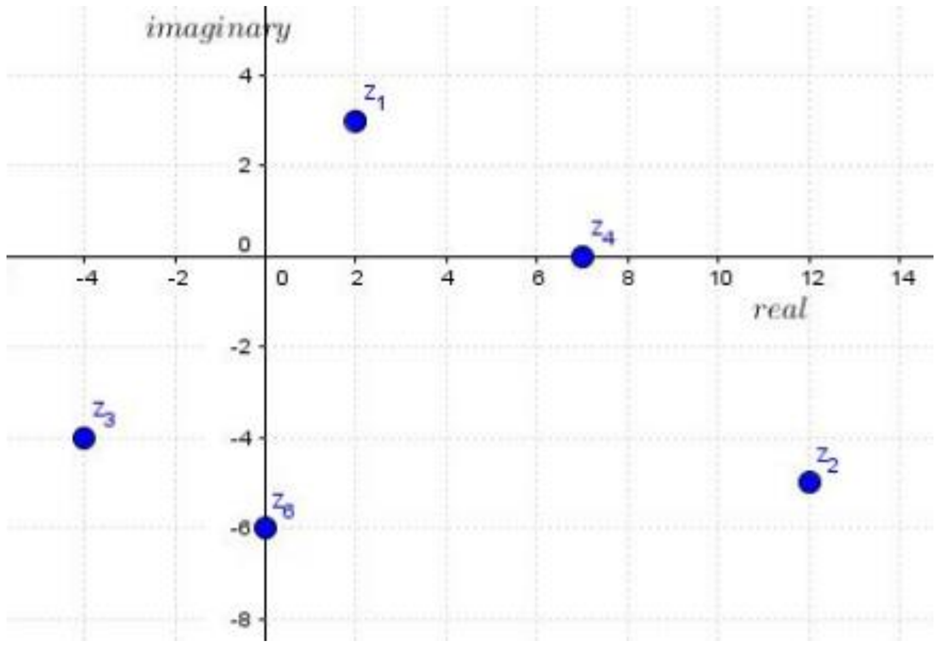


Figure 6.5

Polar form of a Complex number

Using the argand diagram the complex number can be perceived to be vector in which a right triangle is from the (x, y) co-ordinate of the real and imaginary axis. Figure 6.6 below illustrate the relationship of the complex number vector and right triangle.

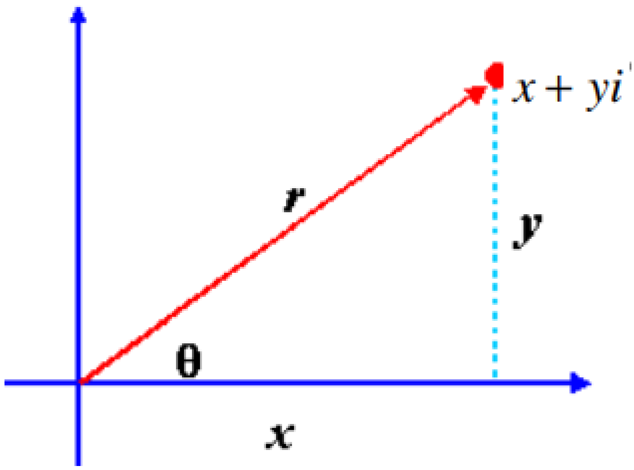


Figure 6.6

From the figure θ is the angle between x-axis real axis and the complex number.

Using trigonometric ratios.

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}$$

Making x and y the subject of the formula we obtain

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Substituting x and y in the complex number.

$z = x + iy$

$z = r\cos\theta + ir\sin\theta$ factorising r

$z = r(\cos\theta + i\sin\theta)$

The expression $z = r(\cos\theta + i\sin\theta)$ is called the **polar form** of complex number z where r- is called the **modulus** and θ is called the **argument.**

From Pythagoras theorem.

$r^2 = x^2 + y^2$

The modulus (z) can also be written as$[z]$. The argument θ can also be calculated from tan $\theta = \frac{y}{x}$ and hence $\theta = \tan^{-1}(y/x)$.

Note the angle obtained in the expression is an acute angle. Using the knowledge of trigonometry it is possible to obtain the argument from its quadrant.

Example 1

Given the complex number $z = 2 + 2i$ find,

a) The modulus of the complex number.
b) The argument of the number.
c) The polar form of the number.

Solution

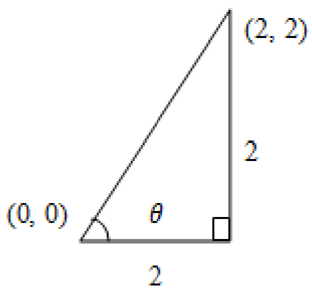
Consider general case: $z = x + yi$

Polar form: $z = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Using



$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ this is the modulus $\tan\theta = \frac{2}{2}$

$\theta = \tan^{-1} 1 = 45^0 = \frac{\pi}{4}$ this is the argument in the interval of $\left[0, \frac{\pi}{2}\right]$

The polar form is hence, $z = 2\sqrt{2}\,(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4})$

Exercise 6.2

Work in groups.

1. Given the complex number $z = {}^-1 + \sqrt{3}\, i$
   - i) Determine the modulus of the complex number z
   - ii) The argument of z
   - iii) The polar form of z

2. Find the polar form of the following complex
   - i) $z_1 = 1 + i$
   - ii) $z_2 = -3 - 3\,i$
   - iii) $z_3 = -i$
   - iv) $z_4 = -3 + i$
   - v) $z_5 = 1 - 2i$

3. A complex number $z_3 = z_1 + z_2$ where $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$ write z_3 in polar form.
4. Given that $z_6 = z_4 . z_5$ and $z_4 = 3i, z_5 = 5$

De Moivre's Theorem

De Moivres theorem is a theory that relates the polar complex number powers to multiply of angles. The theory is named after a French mathematician who was pioneer in the development of analytic trigonometry and in the theorem of probability.

The figure 6.7 alongside shows an image of Abraham de Moivre (Born 26th May, 1667 and died Nov 27, 1754 in London)

Figure 6.7

De Moivre's theorem is importantly used in deriving trigonometric identities and obtaining complex roots of polynomial equation. De Moivre's theorem is derived from the binomial expansion and other trigonometric identities. Consider a complex number $z = r\ (\cos\theta + i\ \sin\theta)$ if z is on a unit circle r= 1 and hence

$z = (\cos\theta + i\ \sin\theta)$

Example 1

Given the De Moiver's equation for a complex number z^n is

$z = (\cos\theta + i\ \sin\theta)$,

i. find the value of;
   a. z^2
   b. z^3
   c. z^4

ii. deduce the general equation for z^n (De Moiver's Theorem)

Solution

a. $z = (\cos\theta + i\ \sin\theta)$

$z^2 = (\cos\theta + i\ \sin\theta)^2 = (\cos\theta + i\ \sin\theta)(\cos\theta + (\sin\theta)$

$= \cos^2\theta + i\ \cos\theta sin\,\theta + cos\theta\ sin\theta + (i)^2 \sin^2\theta$

$= \cos^2\theta + 2i\cos\theta sin\,\theta - \sin^2\theta$, But $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$=\sin^2\theta + 2i\ cos\theta \sin\theta - \sin^2\theta$

$=1 - \sin^2\theta + 2i\ cos\theta \sin\theta - \sin^2\theta$

$$= 1 - 2\sin^2\theta + 2i\ cos\theta\sin\theta$$

But from trigonometric identities, $1 - 2\sin^2\theta = \cos 2\theta$, Substituting this,

$= \cos\theta + 2i\cos\theta sin\theta$

But, $2\ cos\theta \sin\theta = \sin 2\theta$ as an identity, substituting back

$= \cos 2\theta + i\sin 2\theta$

Hence, $z^2 = (\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$

b. $z^3 = z \cdot z^2$

$= (\cos\theta + i\sin\theta)^3 = ((\cos\theta + isin\theta)(\cos\theta + i\,sin\theta)^2\ )$, Since $x^3 = x.\ x^2$

But $(\cos\theta + i\sin\theta) = \cos 2\theta + i\sin 2\,\theta\ from$ z_2 above. ***Substituting***

$= (\cos\theta cos2\theta + icos\theta sin2\theta + i\ sin\theta cos2\theta - sin\theta sin2\theta \ldots (1)$

<u>But</u> cos (A+B) = cos A cos B –sin A sin B is a trigonometric identity

Cos A cos B =cos (A+B) + sin A sin B

$\cos\theta cos2\theta = cos3\theta\theta + sin\theta sin2\theta \ldots (2)$

sin (A+B)= sin A cos B +cos A sin B

cos A cos B = cos A sin B-sin(A+B)

$\cos\theta cos2\theta = cos\theta sin2\theta - sin3\theta \ldots (3)$, Substituting (2) and (3) in (1)

$= \cos 3\theta + sin\theta sin2\theta + icos\theta sin2\theta + 1sin\theta cos2\theta\theta - sin\theta\theta sin2\theta$

$= \cos 3\theta + icos\theta sin2\theta + isin\theta cos2\theta$

But, sin(A+B)=sin A cos B + sin B cos A

$= \cos 3\theta + i(cos\theta + sin2\theta + \sin(2\theta)cos2\theta)$,

Substituting sin(A+B)= sinA cos B+sin B cos A

$\therefore cos\theta sin2\theta + sin2\theta cos\theta = sin3\theta$

$= \cos\theta + isin3\theta$

Hence , z^3 **=** $(\cos\theta + isin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$z^3 = \cos 3\theta + i\sin 3\theta$

c. **note**

$$z^2 = (\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$$
$$z^3 = \cos 3\theta + i\sin 3\theta$$

Hence, $z^4 = (\cos\theta + i sin\theta)^4 = \cos 4x + i\sin 4\theta$

iii. You notice that,

$z = \cos\theta + i\, sin\theta$

$z^2 = (\cos\theta + i\, sin\theta)^2 = cos2\theta + i\, sin2\theta$

$z^3 = (\cos\theta + i\, sin\theta)^3 = cos3\theta + i\, sin3\theta$

$z^4 = (\cos\theta + i\, sin\theta)^4 = cos4\theta + i\, sin4\theta$

In general

$$\boxed{z^n = (\cos\theta + i\, sin\theta)^n = cos4\theta + i\, sin4\theta}$$

The expression is called De Moivre's theorem. It is used to derive trigonometric identities with powers and product an angles and determine roots of complex numbers.

Using De Moivre's Theorem to derive trigonometric identities

Example 2

In group of three students, use De Moivre's Theorem to express $\cos 3\theta$ in terms of $\cos\theta$.

Solution

Recall

$z^3 = (\cos\theta + i\, sin\theta)^3 = cos3\theta + i\, sin3\theta$

Given that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^2$

$$(\cos\theta + i\, sin\theta)^3 = cos^3\theta + 3i\, cos^2\theta sin\theta - 3cosxsin^2\theta - i\, sin^3\theta$$
$$= cos3\theta + i\, sin^3\theta$$

$$cos^3\theta + 3icos^2\theta sin\theta - 3cosxsin^3\theta = cos3\theta - i\, sin3\theta$$

Equating real parts together

$$cos^3\theta - 3icos^2\theta sin\theta - i(cos^2\theta sin\theta - sin^3\theta) = cos3\theta - i\, sin3\theta$$

Since: R.H.S = L.H.S

Real part equal real part complex and complex equal complex part.

$\therefore\ cos^3\theta - 3cos\theta sin^2\theta = \cos 3\,\theta$and but $sin^2\theta = 1 - cos^2\theta$

$=cos^3\theta - 3cos\theta(1 - cos^2\theta)$

$=cos^3\theta - 3cos\theta + 3cos^3\theta$

$=4\ cos^3\theta - 3cos\theta$

Exercise 6.3

Work in groups.

1. Using de Moivre's theorem show that $sin\,3\theta = 3\ cos\ 2\theta sin\theta - sin3\theta$.
2. Use de Moivre's theorem to find the value of $\cos\,4\theta$.
3. Find the value of sinsin 3θ $\ in\ terms\ of\ powers\ of\ sin\theta$.

De Moivre's Theorem and Root of complex numbers

Consider the following questions

Solve a) $8^{1/3}$

b) $25^{1/2}$

c) $(-36)^{1/2}$

Note in the above questions we seek to find a value z satisfy the roots hence

$81^{/3} = z$

$-25^{-1/2} = z$

$(-36)^{1/2} = z$

Eliminating roots on both sides.

$8 = z^3$

$25 = z^2$

$-36 = z^2$

Hence $z^3 - 8 = 0$

$z^2 + 25 = 0$

$z^2 - 36 = 0$

to find the value of z we need to find root of a complex numbers.

The multiplicity of the solution come from the argument of z for any complex number. $z = r(\cos\theta + isin\theta)$

From trigonometry you notice that, the value of θ used is the acute value it is in the interval $\left[0, \frac{\Pi}{2}\right]$. The angle θ can repeat itself after every revolution. Since a revolution after $2\Pi^c$ the angle θ is hence equivalent to $\theta = ,\theta + 2\pi k$, where k=number of revolutions; k= 0, 1,2,3,4,….. for instance ,

$\theta = \theta_{1,}\theta_2 = \theta + 4\boldsymbol{\pi},\ \ \theta_3 = \theta + 6\boldsymbol{\pi}$**,...**

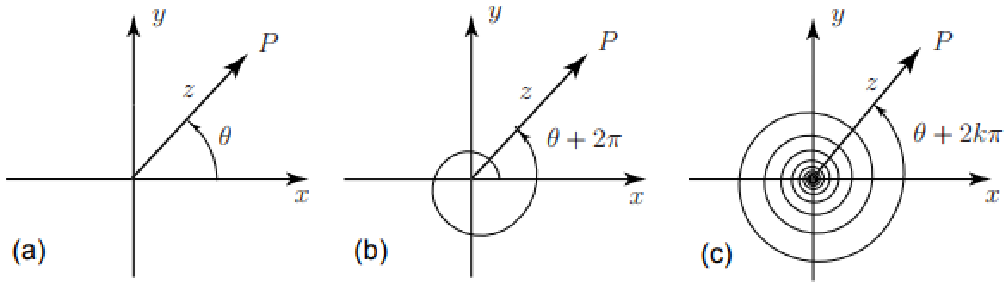


Figure 6.8

Though fig (a) has argument $\theta, (b$ has argument $\theta + 2\boldsymbol{\pi}$and (c) has argument $\theta + 2k\boldsymbol{\pi}$despite the fact that they illustrate same complex number z with r=p

In general

A complex number

z=r $(\cos\theta + i\ sin\theta) = r(\cos(\theta + 2k\boldsymbol{\pi}) +\ i\sin(\theta + 2k\boldsymbol{\pi})$

The De Moivre theorem applies similarities.

$z^n = r^n\ (\cos\theta + i\ sin\theta) = r^n(\cos\ (\theta + 2k\pi) + isinn(n + 2k\pi)$

OR

$Z^{½} = r^{1/n}\ (\cos\frac{\theta}{n}, i\ sin\frac{\theta}{n}) = r^{1/n}\ (\cos\frac{\theta+2k\pi}{n}$ + i sin $(\theta + 2k\boldsymbol{\pi})$

For k= 1, 2, 3…

The number $m^{1/n}$has n-different roots to solve for m-roots we need to find all the values of z at $(k = 0, 1,2,3, , , , , , n)$ k= 0, $\pm$1, 2$\pm$,,,n beyond n the values start repeating themselves.

Example 1

In groups of three, find all the roots of $8^{1/3}$

Solution

Let $z = 8^{1/3}$

$$z^3 = 8$$

Expressing 8 as a complex number (z)

$$8 = 8 + 0i, \text{ hence } z^3 = 8 + 0i$$

$$r = \sqrt{8^2 + o^2 = 8}$$

Expressing z_2 in polar form.

$$z^3 = 8(\cos 2k\pi + i\sin 2k\boldsymbol{\pi}$$

Taking root on both side.

In form of z_2= r $(\cos\theta + i\sin\theta)$

$$\theta = 0^0$$

$$r = \sqrt{8^2 + 0^2 = 8}$$

Expressing z_2 in polar form.

$$z^3 = 8\cos\theta + i\sin\theta)$$

Equating $z = z_2$

$$z^3 = 8(8\cos\theta + i\sin\theta)$$

Using general term of argument $\theta = \theta + 2k\pi$

$$z^3 = 8(\cos 2k\pi + i\sin 2k\pi)$$

Taking root on both sides

$$z = \sqrt[3]{8}\,(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3})$$

$$= z = 2\,(\cos\frac{2k\pi}{3} + i\ \sin\ \frac{2k\pi}{3})$$

Since there are only 3 roots use k=0, 1, 2

At k=0 $\qquad z_0 = 2\,(\cos 0 + I \sin 0^0)$

$$=0$$

At k=1 $\qquad z_1 = 2\,(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$$= 2\cos\frac{2\pi}{3} + 2i\sin\frac{2\pi}{3}$$

$$= 2 \times (-1/2) + 2i\left(\frac{\sqrt{3}}{2}\right)$$

$$= -1 + i\sqrt{3}$$

At k=2 $\qquad z_2 = 2\,(\cos\left(\frac{4\pi}{3}\right) + i\sin\frac{4\pi}{3})$

$$= 2.(-\tfrac{1}{2}) - 2i\left(\frac{\sqrt{3}}{2}\right)$$

$$= -1 - i\sqrt{3}$$

If one try for k=3,,, the solution z_1, z_2 and z_3 start repeating themselves. Hence the root for 81/3 are 2, -1, + i$\sqrt{3}$ and -1-i$\sqrt{3}$.

Note the complex numbers can also be represented as

$$z = [r, \theta] \text{ OR}$$

$$z = [\cos\theta + \sin\theta] \qquad \text{OR}$$

$z = re^{\theta i}$ lead as e exponential θi

where$z \rightarrow$ *complex number*

r$\rightarrow$ $[z] \rightarrow modulus\ \sqrt{x^2 + y^2}$

$\theta \rightarrow argument$

Alternatively, to solve roots of 8 1/3 we say. Let

$z = 8^{\ 1/3}$

$z_3 = 8$

$z_1 = 8$

$z_1 = 8e^{0i}$

In general form

$z_1 = 8e^{(o+2k\pi)i}$

$z^3 = \sqrt{8\ e}\left(\frac{0+2k\pi}{3}\right)i$

$z^{\ 1/3} = \sqrt{8}\,e^{\left(\frac{0+2k\pi}{3}\right)i}$

$z^{\ 1/3} = 2e^{\frac{2k\pi}{3}i}$

Where k= 0, 1, 2,,,,

At k=0, $z_0 = 2e^{oi} = 2$

K=1 $z_1 = 2e^{\frac{12\pi}{3}} = -1 + i\sqrt{3}$

k=2 $z_2 = 2e^{\frac{14\pi}{2}} = -1 + i\sqrt{3}$

Further reading activity

In groups, find all the root of 8 $^{1/3}$ using Argand diagram.

Exercise 6.4

Work in pairs.

1. Using De Moivre's theorem find the value of
   a) $\sin 2\,\theta \; in\ terms\ of \;\; \sin\theta$
   b) $\cos 2\theta \; in\ terms\ of\ cos\theta$

2. Without using table or calculator find $\cos 30^0$ using de Moivre's theorem and $\cos 90^0 = 0$
3. Find all the roots of $(-1)^{1/4}$
4. Find the value of z^{12} given that z= -1 + i$\sqrt{3}$ leave your solution in standard form.
5. Find the 4 complex roots of $z^{1/4}$ given z=1.
6. Express $(\sqrt{3}+i)^7$ in terms of s+bi.
7. Simplify $(\sqrt{2}-i\sqrt{2})^4$
8. Find all the roots of $(\sqrt{3}+i)^{1/5}$
9. Write $(2+i)^3$ in standard form.

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